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ENAMA 2026

ANAIS DO XIX ENAMA

04 a 06 de Novembro 2026

Conteúdo

Closed subspaces of linear and multilinear operators determined by sets and sequences, by G. Botelho & A. Monção 7

Asymptotic analysis for Riemann-Hilbert problems and small gap probabilities for a conditional process, by V. Alves, C. F. Santos, G. L. F. Silva & L. Zhang 9

Generalized Entropy Numbers and ε-Entropy Functions of Sets and Operators, by E. B. da Silva & L. C. D. Pasqua 11

Polynomial numerical index with respect to a norm-one polynomial, by F. J. Bertoloto & E. R. Santos 13

Weak Minimizing Property and Reflexivity, by V. Kadets & G. Ribeiro 15

The Right Topology on Subspaces, by Geraldo Botelho & Yara M. Corrêa 17

Orthogonal projections onto invariant subspaces of the Hardy space over the bidisc, by J. M. R. Do Carmo & M. S. Ferreira 19

Shadowing for weighted composition operators on spaces of continuous functions, by João V. A. Pinto . . 21

Topologies defined by operator classes on Banach spaces, by Geraldo Botelho & Juan David Rubio Tabares 23

Three Different Proofs that the Ratios Between Consecutive Fibonacci Numbers Converge to the Golden Number, by Karen Eduarda Nobokite, Míriam Saldanha Carneiro & Marco Antônio de Alcântara Fernandes 25

Robust structures and cyclic operators, by N. G. Albuquerque 27

The weakly p-singular maximizing property, by Oscar Iván Blanco Ardila & Vinícius Miranda 29

Cyclicity and complex symmetry of composition operators on Paley-Wiener spaces, by P. V. Hai, S.W. Noor & O. R. Severiano 31

Geometric Aspects of the Lipschitz-free Spaces, by R. D. Garcia & V. M. Cortes (supervisor) 33

Copies of ℓ_∞ in the relative complements of the space of holomorphic Lipschitz bi-forms, by R. D. Garcia & V. M. Cortes (supervisor) 35

On Properties of the Pisier-Xu Space, by Willian H. G. Corrêa & Wilson A. Cuellar 37

Mathematical Modeling of Peptide Formation under Prebiotic Conditions, by Bianca dos Santos, Dimas Augusto Morozin Zaia, Eliandro Rodrigues Cirilo & Paulo Laerte Natti 39

Constructing Positivity-Preserving Integrators for Stochastic differential equations, by H. de la Cruz & M. Munoz 41

Neutron spectral reconstruction via deep learning and physics-informed neural networks a comparative analysis, by I. Deme, M. Sequeiros & A. Caminha 43

Extreme point maximization of convex quadratic response surfaces and exact leave-one-out validation, by T. F. Oliveira, V. H . S. Ramos & I. I. Barbosa 45

Controlling the Condition Number of Multiquadric RBF Matrices via Poisson Disk Sampling, by João Rogério da Silva 47

Optimization of the polymer injection process for improved oil recovery, by P. N. Moraes & B. M. Rocha 49

Fractional SIR model with cross diffusion, by S. R. Mazorche & T. F. C. Santos 51

On growth rate comparisons: μ-dichotomy, bounded growth, and spectral properties, by C. A. Gallegos & N. Jara 53

On stability for linear measure functional differential equations through generalized linear equations, by F. Andrade da Silva, C. Carrasco & C. A. Gallegos 55

Topological Methods in the Study of the Existence of at Least One Pair of Nonnegative Periodic Solutions for a Particular Class of Ordinary Differential Equations, by Pierluigi Benevieri, Juan Francisco Camasca Fernández & Guglielmo Feltrin 57

On (θ, T)-periodic solutions and Floquet theory for abstract generalized ODEs, by M. Ap. Silva 59

A comprehensive family of memory kernels generated by the gamma Mittag-Leffler distribution, by N. Z. Monteiro, S. R. Mazorche & R. W. dos Santos 61

(ω, c)-periodicity on isolated time scales, by Marielle Ap. Silva, Martin Bohner, Jaqueline G. Mesquita & Ronaldo Murakami 63

Critical double phase equations in $\mathbb{R}^N$ with logarithmic nonlinearities: what's new?, by A. Bahrouni, A. Fiscella & P. Winkert 65

Existence of Solutions for a Semilinear Elliptic Problem via Direct Method and the Mountain Pass Theorem, by A. P. Rodrigues & A. H. S. Medeiros 67

On a special class of nonlocal Neumann problems with critical growth, by A. C. R Costa & F. B. Silva 69

On the Existence, Regularizing Effects, and Asymptotic Behavior of (p, q)-Laplacian Schrödinger-Maxwell Systems, by Anderson L. A. de Araujo, Ayana P. de Castro Santana & Luiz F. O. Faria 71

FitzHugh-Nagumo-Type Systems with Sign-Changing Coefficients, by J. M. do Ó, E. Shamarova & V. V. Silva 73

Fractional Hardy-Hénon Equations with Nonlinearities of Critical Exponential Growth, by E. M. Barboza, O. H. Miyagaki, F. R. Pereira & C. R. Santana 75

A convective 1-homogeneous p-Laplacian problem, by G. Ercole 77

Existence and concentration solutions for a coupled elliptic systems with critical exponential growth in $\mathbb{R}^2$ and potentials well, by G. S. A. Costa, G. M. Figueiredo & S. I. Moreira 79

Symmetry and classification of positive standing waves of nonlinear Hartree type equations, by E. de S. Böer, E. M. dos Santos & G. de P. Ramos 81

Critical Schrödinger-Bopp-Podolsky Systems: Solutions in the Semiclassical limit, by H.M. Santos Damian & G. Siciliano 83

Singular Elliptic Problems Involving the 1-Laplacian with Unbounded Superlinear Terms, by L. F. O. Faria, A. Gonzaga, J. C. Silva & E. D. Silva 85

Kirchhoff p-Laplacian equation with a nonlocal singular exponent, by J. Carbajal Licas & E. Cabanillas Lapa 87

Study of Singular Elliptic Problems Involving the p-Laplacian, by K. E. Nobokite & G. A. Pereira 89

Schauder estimates for non-convex fully nonlinear elliptic PDEs with Dini continuous data, by Junior da Silva Bessa, João Vitor da Silva & Laura A. Ospina C. 91

Nonnegative solutions to a weighted indefinite elliptic Schrödinger equation, by Gustavo S. A. Costa, Lucas G. F. Cunha & José C. O. Junior 93

Multiplicity Results for subcritical Choquard-Type Superlinear Elliptic Problems with Partial Interference with the Spectrum, by L. A. Miñope G. & F. O. Vieira de Paiva 95

Optimal gradient estimates for quasilinear elliptic models of p−Laplacian type via blow-up procedure, by Luis Carlos Urbiñes Suarez & João Vitor da Silva 97

Sharp regularity for the obstacle problem for quasilinear elliptic equations of p−Laplacian type, by E. C. Bezerra Júnior, J. V. da Silva & R. T. Frias 99

A Quasilinear Elliptic Equation Beyond the Trudinger-Moser Growth, by A. L. A. de Araujo, A. H. S. Medeiros, J. A. L. Tordecilla & S. M. Valério 101

Critical Exponential Growth in Quasilinear Elliptic Problems: A Nehari Manifold Approach, by G. Figueiredo, S. Moreira & R. Ruviaro 103

Existence and multiplicity of solutions for a critical system involving a biharmonic and p-Laplacian operator, by R. D. Carlos & S. M. A. Salirrosas 105

A Subdifferential Approach to the $p \to 1^+$ Limit of Fractional Sobolev Constants, by G. A. Pereira, G. Ercole & T. V. O. Alves 107

Multiplicity of Solutions for a Stationary Schrödinger Equation via Lusternik-Schnirelmann Theory, by V. C. Santos & E. Toon 109

Nonlinear fractional magnetic problem with an arbitrary growth at infinity, by W. F. Almeida & G. M. Figueiredo 111

A Nonlinear Parabolic Problem Involving a Kirchhoff-Type $p(b(u))\&q(b(u))$-Operator, by W.Barahona M., E. Cabanillas L, R. De La Cruz M. & J. Luque R. 113

Dynamics of a nonlinear swelling porous thermoelastic system with second sound, by Alberto L. C. Costa, Flank D. M. Bezerra & Ronal Q. Caljaro 115

Extensible plate equation with thermal memory and exponential perturbations, by A. R. A. Barbosa & M. A. Jorge Silva 117

Mathematical Modeling of Particle Transport in Porous Media under Rainfall-Induced Flow, by A. M. Neves, J. Zoffoli Caçador, G. Dutra Sousa & G. Chapiro 119

Inviscid Damping and Linear Chaos for the Rayleigh Equation using C_0-Semigroups, by C. J. Medeiros . 121

Decay rates of C_0-semigroup: Mlog theorems (Batty-Duyckaerts) and optimality theorems (Borichev-Tomilov), by D. N. F. Silva & G. S. Santana 123

Critical blow-up for a shifted Tricomi equation with an oscillatory scale-invariant mass, by D. Marcon, W. Nascimento & M. Santos 125

Dimensional reduction and numerical approximation of an inhomogeneous Stokes model in a rectangular channel, by Edward D. Gonzalez Andrés Julián Castrillón Vásquez & Juan Bolagay 127

Analytical modeling of foam-assisted displacement with surfactant adsorption in porous media, by Grigori Chapiro, Pavel Bedrikovetski & Giulia C. Fritis 129

Robustness of Non-Autonomous Attractors for Coupled Pseudo-Parabolic Systems in $\mathbb{R}^n$, by Mirelson M. Freitas, Italo A. O. de Albuquerque & Baowei Feng 131

Asymptotic behavior of a semilinear wave equation with both localized memory and delay effects, by J. C. O. Faria, C. A. Okawa & A. Y. Souza Franco 133

Boundary and Interior Control in a Diffusive Lotka-Volterra Model, by J. C. Barreira, M. Sonego & E. Zuazua 135

Weakly Nonlinear Models for Hydroelastic Water Waves, by D. Alonso-Orán, R. Granero-Belinchón & J. S. Ziebell 137

Global strong solutions and decay estimates for a penalized nonhomogeneous nematic liquid crystal model in 3D thin domains, by L. E. Mascarenhas-Silva & M. A. Rojas-Medar 139

Blow-Up Prevention in a Predator-Prey Model with Singular Sensitivity, by B. Telch & L. G. Longen . 141

Boundary null controllability of degenerate heat equation as the limit of internal controllability, by B. S. V. Araújo, R. Demarque & L. Viana 143

Intermediate asymptotics and sharp transition rates for the viscous Burgers equation, by M. Sonego & E. Zuazua 145

Blow-up for a semilinear Tricomi-type equation with scale-invariant mass in the oscillatory regime, by D. Marcon, W. N. do Nascimento & M. C. Santos 147

Upper and Lower Estimates for Derivatives of Solutions to a Class of Advection-Diffusion Equations, by P. L. Guidolin, L. Schutz & J. S. Ziebell 149

Lamé system with nonlinear vector field under critical exponent, by P. F. M. Damazio, M. A. Jorge Silva & R. G. Camara 151

Regularity for solutions of degenerate wave equation with normal derivative in L^2, by Bruno S. V. Araújo, Reginaldo Demarque & Luiz Viana 153

Pseudo-differential homogenization procedure in the composite materials., by S. Sergeev 155

Rainfall-Triggered Slope Liquefaction Modeling in Juiz de Fora via Hyperbolic Conservation Laws, by P. H. M. Azevedo, L. L. Lanna, T. S. Oliveira & G. Chapiro 157

Existence and continuous dependence of the local or global solution for nonhomogeneous heat equations, by Y. S. S. Ayala 159

ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *7–8*

CLOSED SUBSPACES OF LINEAR AND MULTILINEAR OPERATORS DETERMINED BY SETS AND SEQUENCES

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Abstract

We show that classes of linear operators that map bounded sets satisfying a cone property to relatively compact sets are *always* closed subspaces. Furthermore, we investigate spaces of multilinear operators defined or characterized by their action on bounded sequences with special properties.

1 Introduction

Let $E, E_1, \ldots, E_m, F$ be Banach spaces. Denote by $\mathcal{L}(E,F)$ the Banach space of bounded linear operators from E to F, endowed with its usual norm, and by $\mathcal{L}(E_1,\ldots,E_m;F)$ the Banach space of continuous m-linear operators from $E_1 \times \cdots \times E_m$ to F, also endowed with its usual norm. Closed subspaces of these operator spaces play a key role in Banach space theory, particularly in the study of classes of linear and multilinear operators with special properties. Proofs that different subspaces of operators are closed often rely on essentially the same argument. Our aim is to avoid, or at least substantially reduce, such unnecessary repetition.

Some terminology is needed. The following concepts can be found in [2, 4, 5]:

• A *topological vector space* is a vector space endowed with a topology for which vector addition and scalar multiplication are continuous.

We say that a topological vector space E is:

• *Quasi-complete* if every bounded closed subset of E is complete with respect to the uniform structure induced by E [5, 1.4, p. 16].

• *Locally convex* if the origin has a neighborhood basis consisting of convex sets, or equivalently, if its topology is generated by a family of seminorms [2, Corollary 8.3.8]. In what follows, all locally convex spaces are assumed to be Hausdorff.

A subset A of E is said to be:

∘ *Totally bounded* if, for every neighborhood V of the origin, there are $a_1,\ldots,a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n}(a_i+V)$.

∘ *Relatively compact* if the closure $\overline{A}$ of A in E is compact.

Let E be a normed space and let F be a locally convex space whose topology is generated by a family of seminorms $\{p_i : i \in I\}$. The *topology of uniform convergence on bounded sets* on the space $\mathcal{L}(E,F)$ of continuous linear operators from E to F is the topology generated by the family of seminorms $\{P_i : i \in I\}$, where, for each $i \in I$, $P_i(T) = \sup\{p_i(T(x)) : \|x\| \leq 1\}$ for every $T \in \mathcal{L}(E,F)$. When F is also a normed space, this topology coincides with the usual norm topology on $\mathcal{L}(E,F)$ [5, p. 81].

2 Main Results

We say that a property $\mathfrak{P}$ of bounded subsets of a given normed space is a *cone property* if λC has $\mathfrak{P}$ whenever $\lambda > 0$ and C has $\mathfrak{P}$.

Theorem 2.1. *Let E be a normed space, let F be a quasi-complete locally convex space, and let $\mathfrak{P}$ be a cone property of bounded subsets of E. Then the set of operators $T \in \mathcal{L}(E,F)$ that map bounded subsets of E having $\mathfrak{P}$ to relatively compact subsets of F is a closed subspace of $\mathcal{L}(E,F)$ in the topology of uniform convergence on bounded sets.*

Let $E_1, \ldots, E_m$ be Banach spaces. We say that a sequence $((x_1^n, \ldots, x_m^n))_n$ in $E_1 \times \cdots \times E_m$ is *m-bounded* if $\sup_n \|x_1^n\| \cdots \|x_m^n\| < \infty$. Of course, bounded sequences are m-bounded.

Let $\mathfrak{R}$ denote the subset of $(E_1 \times \cdots \times E_m)^{\mathbb{N}}$ consisting of all m-bounded sequences. Let $\sim$ be a relation from $\mathfrak{R}$ to $E_1 \times \cdots \times E_m$; that is, $\sim$ is a subset of $\mathfrak{R} \times (E_1 \times \cdots \times E_m)$. When the ordered pair $(((x_1^n, \ldots, x_m^n))_n, (x_1, \ldots, x_m))$ belongs to $\sim$, we write $((x_1^n, \ldots, x_m^n))_n \sim (x_1, \ldots, x_m)$.

With the notation and terminology introduced above, we have the following:

Proposition 2.1. *Let $E_1, \ldots, E_m$ be Banach spaces. For every Banach space F, the set of multilinear operators $A \in \mathcal{L}(E_1, \ldots, E_m; F)$ such that $A(x_1^n, \ldots, x_m^n) \longrightarrow A(x_1, \ldots, x_m)$ in F whenever the sequence $((x_1^n, \ldots, x_m^n))_n$ is m-bounded and $((x_1^n, \ldots, x_m^n))_n \sim (x_1, \ldots, x_m)$ is a closed subspace of $\mathcal{L}(E_1, \ldots, E_m; F)$.*

Corollary 2.1. *Let $E_1, \ldots, E_m, F$ be Banach spaces and, for each $k = 1, \ldots, m$, let τ_k be a (not necessarily vector) topology on E_k such that τ_k-convergent sequences in E_k are bounded. Then the set of operators $A \in \mathcal{L}(E_1, \ldots, E_m; F)$ such that $A(x_1^n, \ldots, x_m^n) \longrightarrow A(x_1, \ldots, x_m)$ in F whenever $x_k^n \xrightarrow{\tau_k} x_k$ in E_k for each $k = 1, \ldots, m$ is a closed subspace of $\mathcal{L}(E_1, \ldots, E_m; F)$.*

Proof. Since bounded sequences in $E_1 \times \cdots \times E_m$ are m-bounded, it is enough to define

$$((x_1^n, \ldots, x_m^n))_n \sim (x_1, \ldots, x_m) \Longleftrightarrow x_1^n \xrightarrow{\tau_1} x_1, \ \ldots, x_m^n \xrightarrow{\tau_m} x_m$$

and apply Proposition 2.1. □

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XIX ENAMA - Novembro 2026 *9–10*

ASYMPTOTIC ANALYSIS FOR RIEMANN-HILBERT PROBLEMS AND SMALL GAP PROBABILITIES FOR A CONDITIONAL PROCESS

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Abstract

This work uses asymptotic analysis of Riemann-Hilbert boundary value problems in order to study a point process coming from random matrix theory. Our main results are asymptotic formulas for the probability of the process to have only points to the left of some threshold as this threshold goes to $-\infty$.

1 Introduction

Riemann-Hilbert problems (RHPs, for short) have been used with great success in the past few decades to analyse a number of special functions and integrable systems. An RHP is the problem of finding a function, typically matrix valued, which is analytic on the complex plane except for a contour, subject to: a jump condition for its boundary limits at the contour; a prescribed asymptotic behaviour at infinity and; growth condition at endpoints or singular points of the contour. When the an RHP depends on a parameter t, asymptotics as, say, $t \to \infty$ can be obtained systematically by the method of nonlinear steepest descent, introduced by Deift and Zhou in [2].

This work aims to study gap probabilities for a particular point process via Deift and Zhou's method. The Airy_2 process appears as a limit point process for rescaled eigenvalues of Hermitian random matrices. Let $\sigma : \mathbb{R} \to [0,1]$, the *thinning function*, $s \in \mathbb{R}$ and $t > 0$, which are shifting and scaling parameter. Given a realization of Airy_2, remove a point at a position x with probability $1 - \sigma(tx - s)$. Now, conditioning on the event that no particle was lost we get the *conditional thinning* of Airy_2. This is a determinantal point process whose correlation kernel [1, 3] is built upon a function $\Phi(\zeta) = \Phi(\zeta \mid S, T)$ satisfying

$$\partial_S^2 \Phi(\zeta \mid S,T) = \left[\zeta + 2\frac{S}{2T} - \frac{1}{T}\int_{\mathbb{R}} \Phi(x;S,T)^2 \,\mathrm{d}\sigma(x)\right] \Phi(\zeta \mid S,T)$$

with asymptotic behaviour

$$\Phi(\zeta \mid S,T) \sim T^{1/6}\mathrm{Ai}\left(T^{2/3}\zeta + ST^{-1/3}\right), \quad \zeta \to +\infty \text{ with } |\arg\zeta| < \pi - \theta$$

for any $\theta \in (0,\pi)$ fixed. Here $T = t^{-3/2}$, $S = st^{-3/2}$ and Ai is the Airy function. We choose $\sigma(x) = \dfrac{1}{1+e^x}$, which has been related to the Kardar-Parisi-Zhang equation, see [1].

Let $F_\sigma(\tau) = F_\sigma(\tau \mid S,T)$ probability is the probability of no particle being greater than τ. Our goal is to provide asymptotic formulas for this quantity when $\tau \to -\infty$, which we call *small gap asymptotics*. A Fredholm determinant characterization of $F_\sigma(\tau)$ and the particular form of the correlation kernel – namely its integrability in the sense of Its-Izergin-Korepin-Slavnov – allows us to use Deift-Zhou's method.

2 Main Results

Our starting point is the following Riemann-Hilbert problem, which is known to be uniquely solvable. Find a 2×2 matrix-valued analytic function $Y : \mathbb{C} \setminus [\tau, +\infty) \to \mathbb{C}^{2\times 2}$ such that:

(i) Y has continuous boundary values $Y_\pm$ along $(\tau, +\infty)$, related by

$$Y_+(x) = \left[I - 2\pi i \sigma(x) \begin{pmatrix} \partial_S \Phi(\zeta \mid S, T) \\ \Phi(\zeta \mid S, T) \end{pmatrix} \begin{pmatrix} -\Phi(\zeta \mid S, T) \\ \partial_S \Phi(\zeta \mid S, T) \end{pmatrix}^T \right] Y_-(x), \quad x > \tau;$$

(ii) Y satisfies growth conditions

$$Y(z) = I + \frac{Y_1}{z} + \mathcal{O}(z^{-2}) \text{ as } z \to \infty \quad \text{and} \quad Y(z) = \mathcal{O}(\log|z - \tau|) \text{ as } z \to \tau$$

where Y_1 is a z-independent matrix.

Proposition 2.1. *Let Y be the solution to the RHP above. Then, $(Y_1)_{21}$, the $(2,1)$-entry of Y_1, equals $-\partial_S \log F_\sigma(\tau, S, T)$.*

The nonlinear steepest descent method for RHPs consists in applying explicit invertible transformations in order to obtain an RHP whose jump is asymptotically close to the identity. In this case the solution itself is shown to exist and be close to the identity, and asymptotics for desired quantities can be deduced tracing back the steps. With this method, we obtained the following results as $\tau \to -\infty$, depending on the new parameter

$$\eta = T^{\frac{2}{3}} \left(\frac{S}{T} - |\tau| \right).$$

Theorem 2.1. *Let $T_0 > 0$ be given.*

(i) *There exists $M_0 > 0$ such that*

$$\log F_\sigma(\tau, S, T) = \mathcal{O}\left(\frac{1}{\eta^3}\right), \quad \textit{uniformly for } \eta > M_0 \textit{ and } 0 < T < T_0.$$

(ii) *Given M_0 as above, for any $M > M_0$ there exists $\tau_0 > 0$ such that*

$$\log F_\sigma(\tau, S, T) = \log F_{TW}(\eta) + \mathcal{O}\left(\frac{1}{M^3}\right) + \mathcal{O}\left(\frac{M}{T^{\frac{2}{3}}|\tau|}\right), \textit{ uniformly for}$$

uniformly for $|\eta| \leq M$, $0 < T < T_0$ and $|\tau| > \tau_0 T^{-\frac{2}{3}}$. Here, F_{TW} the Tracy-Widom distribution.

We found another formula for $-KT^{\frac{2}{3}}|\tau| \leq \eta \leq -M_1$, $0 < T < T_0$ and $|\tau| > \tau_1 T^{-\frac{2}{3}}$, where $K > 1$, $M_1 > M_0$ and $\tau_1 > 0$ are given. It will be omitted due to space constraints.

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XIX ENAMA - Novembro 2026 *11–12*

GENERALIZED ENTROPY NUMBERS AND ε-ENTROPY FUNCTIONS OF SETS AND OPERATORS

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Abstract

In this work, we introduce and study generalized entropy numbers for sets and operators acting on Banach spaces. We also introduce the concept of generalized ε-entropy functions. The classical notion of Hausdorff entropy numbers becomes a particular case of the introduced definition. Several other examples of particular cases of generalized entropy numbers are also given. We study the relationship between entropy numbers and epsilon-entropy of sets. Several properties are demonstrated for the general case.

1 Introduction

The concepts of ε-entropy $\mathcal{H}_\varepsilon(B)$ and ε-capacity $\mathcal{C}_\varepsilon(B)$ of a totally bounded subset B of a metric space M were first described by Pontrjagin and Schnirelmann in [4] and by Kolmogorov in of this topic in [2]. In these works, among other quantities, the ball covering number $\mathcal{N}_\varepsilon(B)$ and the packing number $\mathcal{M}_\varepsilon(B)$ of B were defined, introducing and studying fundamental asymptotic characteristics of subsets of metric spaces.

Using these ideas, Pietsch introduced the entropy numbers of linear operators in [3] and studied their properties. Carl and Stephani also came up with the entropy numbers $\varepsilon_n(B)$ and the capacity (inner entropy) $\varphi_n(B)$ of B. They then studied the properties of these numbers in metric and Banach spaces, as well as their relationships with bounded operators between Banach spaces (see [1]). Entropy numbers become a fundamental tool in the theory of operators and in the local theory of Banach spaces, and it is important in the study of eigenvalue distribution of operators in Banach spaces. On the other hand, ε-entropy was much less studied in the literature.

2 Main Results

We begin by presenting our definition of a generalized entropy number sequence, in accordance with [5].

Definition 2.1. *Let X be a Banach space and $\mathcal{B}$ the family of all bounded sets of X. A sequence of mappings $(u_n)_{n=1}^{\infty}$, where $u_n : \mathcal{B} \to [0, \infty)$, is called a* ***generalized entropy number sequence*** *(GENS) defined on X if the following conditions are satisfied for every $B \in \mathcal{B}$ and every $n \in \mathbb{N}$:*

1. $u_n(B) = u_n(\overline{B})$;
2. $u_1(B) \geq u_2(B) \geq \cdots \geq 0$;
3. *If $B_1 \subset B_2$, then $u_n(B_1) \leq u_n(B_2)$ for all $B_1, B_2 \in \mathcal{B}$;*
4. *There exists a positive constant $M(u)$ such that*

$$u_{mn}\left(\bigcup_{i=1}^{n} B_i\right) \leq M(u) \max_{1 \leq i \leq n} \{u_m(B_i)\},$$

for all $B_i \in \mathcal{B}$ and every $m \in \mathbb{N}$.

5. $u_n(tB) = |t|\, u_n(B)$, *for every* $t \in \mathbb{R}$;

6. $u_n(x + B) = u_n(B)$, *for every* $x \in X$.

Proposition 2.1. *The* ***Kuratowski entropy numbers*** *of a bounded subset* B *of* X*, defined by*

$$\alpha_n(B) = \inf\big\{\varepsilon > 0 : \textit{there exists an } \varepsilon\textit{-covering of } B \textit{ by } q \leq n \textit{ sets}\big\},$$

form a GENS on X.

We also introduce the definition of generalized ε-entropy functions, but first we need to define the HU condition, ([6]).

Definition 2.2. *We say that a family* $\emptyset \neq \mathcal{D}$ *of subsets of* $\mathcal{B}$ *satisfies the HU* ***condition*** *if, for every* $B, B_1, B_2 \in \mathcal{D}$, $x \in X$, *and* $0 < \alpha < 1$, *we have* $B_1 \cup B_2, x + B, \alpha B \in \mathcal{D}$.

Definition 2.3. *Let* X *be a Banach space, let* $\emptyset \neq \mathcal{D} \subset \mathcal{B}$ *satisfy the HU condition, and let* $\lambda_0 \geq 0$. *A function* $f : (\lambda_0, \infty) \times \mathcal{D} \longrightarrow \mathbb{N}$ *is called a* ***generalized*** ε***-entropy function*** *(GEEF) if it satisfies the following properties for all* $B, B_1, B_2 \in \mathcal{D}$, $x \in X$, $0 < \alpha < 1$, *and* $\lambda > \lambda_0$, *where* $f_\lambda(\cdot) = f(\lambda, \cdot)$:

1. *If* $\lambda_0 < \lambda_2 \leq \lambda_1$, *then* $f_{\lambda_1}(B) \leq f_{\lambda_2}(B)$;

2. *If* $B_1 \subset B_2$, *then* $f_\lambda(B_1) \leq f_\lambda(B_2)$;

3. $f_\lambda(B_1 \cup B_2) \leq f_\lambda(B_1) + f_\lambda(B_2)$;

4. $f_\lambda(x + B) = f_\lambda(B)$;

5. $f_\lambda(\alpha B) = f_{\frac{\lambda}{\alpha}}(B)$.

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POLYNOMIAL NUMERICAL INDEX WITH RESPECT TO A NORM-ONE POLYNOMIAL

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Abstract

In this work, we introduce a polynomial numerical index for pairs of Banach spaces with respect to a norm-one k-homogeneous polynomial, extending the numerical index with respect to a norm-one operator to the polynomial setting.

1 Introduction

Let X and Y be Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by X^* the topological dual of X and by S_X the unit sphere of X. The Banach spaces of all bounded linear operators from X into Y and of all continuous k-homogeneous polynomials from X into Y are denoted by $\mathcal{L}(X,Y)$ and $\mathcal{P}(^kX,Y)$, respectively. Given a subset C of X, we denote by $\operatorname{conv} C$ the convex hull of C.

The *numerical index* of a Banach space X is defined by

$$n(X) = \inf\{v(T) : T \in \mathcal{L}(X,X), \|T\| = 1\},$$

where

$$v(T) = \sup\{|x^*(T(x))| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

is called the *numerical radius* of T. The concept of numerical index was first suggested by G. Lumer in 1968.

In 2006, Y. S. Choi et al. [2] introduced the concept of the *polynomial numerical index of order k*. Given $k \in \mathbb{N}$, the polynomial numerical index of order k of a Banach space X is defined as

$$n^{(k)}(X) = \inf\{v(P) : P \in \mathcal{P}(^kX,X), \|P\| = 1\},$$

where

$$v(P) = \sup\{|x^*(P(x))| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

is the *numerical radius* of P.

In 2014, M. A. Ardalani [1] introduced numerical concepts with respect to a norm-one operator. Given $T, G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, the *numerical range of T with respect to G* is defined by

$$V_G(T) = \bigcap_{\delta>0} \overline{\{y^*(Tx) : x \in S_X,\ y^* \in S_{Y^*},\ \operatorname{Re} y^*(Gx) > 1-\delta\}}.$$

The corresponding *numerical radius* and *numerical index* are

$$v_G(T) = \sup\{|\lambda| : \lambda \in V_G(T)\} \quad \text{and} \quad n_G(X,Y) = \inf\{v_G(T) : T \in \mathcal{L}(X,Y),\ \|T\| = 1\}.$$

Motivated by these notions, we introduce below the corresponding polynomial construction for a fixed norm-one homogeneous polynomial [3].

2 The polynomial index

Throughout this section, let X, Y be Banach spaces and let $Q \in \mathcal{P}(^kX, Y)$ with $\|Q\| = 1$. We first introduce the intrinsic version of the numerical range.

Definition 2.1. *For every $P \in \mathcal{P}(^kX, Y)$, the* **intrinsic numerical range of P with respect to Q** *is the set*

$$V(\mathcal{P}(^kX, Y), Q, P) := \{\phi(P) : \phi \in \mathcal{P}(^kX, Y)^*, \|\phi\| = \phi(Q) = 1\}.$$

The spatial version follows the linear theory by using points where Q almost attains its norm.

Definition 2.2. *For every $P \in \mathcal{P}(^kX, Y)$, the* **approximated spatial numerical range of P with respect to Q** *is the set*

$$V_Q(P) := \bigcap_{\delta>0} \overline{\{y^*(P(x)) : y^* \in S_{Y^*},\ x \in S_X,\ \operatorname{Re} y^*(Q(x)) > 1 - \delta\}}.$$

The two ranges are linked as follows.

Proposition 2.1. *For every $P \in \mathcal{P}(^kX, Y)$, we have*

$$V(\mathcal{P}(^kX, Y), Q, P) = \operatorname{conv} V_Q(P).$$

Consequently, the quantities

$$v(\mathcal{P}(^kX, Y), Q, P) = \sup\{|\lambda| : \lambda \in V(\mathcal{P}(^kX, Y), Q, P)\},$$
$$v_Q(P) = \sup\{|\lambda| : \lambda \in V_Q(P)\}$$

coincide. The number $v_Q(P)$ is called the *numerical radius of P with respect to Q*.

This common radius allows us to define the corresponding polynomial index.

Definition 2.3. *The* **polynomial numerical index of (X, Y) with respect to Q** *is the number*

$$n_Q^{(k)}(X, Y) = \inf\{v_Q(P) : P \in \mathcal{P}(^kX, Y), \|P\| = 1\}.$$

When $k = 1$, this definition recovers the concept of the numerical index of (X, Y) with respect to a norm-one operator, as introduced in [1]. Further properties and examples of this index will be discussed in the presentation.

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WEAK MINIMIZING PROPERTY AND REFLEXIVITY

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Abstract

For a bounded linear operator $T : X \longrightarrow Y$, put $m(T) = \inf\{\|Tx\| : x \in S_X\}$. A sequence (x_n) in S_X is minimizing for T if $\|Tx_n\| \longrightarrow m(T)$. The pair (X, Y) has the weak minimizing property if every operator admitting a non-weakly null minimizing sequence attains its minimum on S_X. We show that, when Y is infinite-dimensional and X has a countable total set of functionals, this property forces X to be reflexive. We also describe the reflexive case in terms of whether Y contains an isomorphic copy of X.

1 Introduction

Norm attainment and minimum norm attainment are natural optimization problems for operators between Banach spaces, but their behavior differs substantially: every non-injective operator attains its minimum, which is then zero. Motivated by the weak maximizing property, Chakraborty introduced the following notion [1].

Definition 1.1. *A pair (X, Y) of Banach spaces has the* weak minimizing property *if every $T \in L(X, Y)$ having a minimizing sequence (x_n) which is not weakly null is min-attaining, that is, $\|Tx\| = m(T)$ for some $x \in S_X$.*

This property is also relevant to operator approximation: it implies that the min-attaining operators are dense in $L(X, Y)$ [2]. Our aim is to show that, for a large class of spaces, it detects reflexivity of the domain. This parallels James' characterization of reflexivity through norm-attaining functionals [3], although the proof requires a different mechanism.

2 Main Results

Theorem 2.1. *Let X be a non-reflexive Banach space possessing a countable total set of functionals, and let Y be infinite-dimensional. Then (X, Y) does not have the weak minimizing property. Consequently, if X is separable, Y is infinite-dimensional and (X, Y) has the weak minimizing property, then X is reflexive.*

Proof in the separable case. There is a biorthogonal system $(u_n, u_n^*)_{n\geq 1} \subset S_X \times X^*$ such that (u_n) is complete, (u_n^*) is total, and (u_n) is not weakly null. Indeed, one combines the Milman–Milman characterization of non-reflexivity with the existence of a Markushevich basis containing a prescribed basic sequence. Let (v_n) be a normalized basic sequence in Y and define

$$Tx = \sum_{n=1}^{\infty} \frac{u_n^*(x)}{2^n \|u_n^*\|} v_n. \tag{1}$$

The series is absolutely convergent and $\|T\| \leq 1$. Moreover, $Tu_n = v_n/(2^n\|u_n^*\|)$, so $\|Tu_n\| \longrightarrow 0$. Hence $m(T) = 0$ and (u_n) is a non-weakly null minimizing sequence. On the other hand, T is injective: if $Tx = 0$, uniqueness of the coefficients of the basic sequence gives $u_n^*(x) = 0$ for every n, and totality yields $x = 0$. Therefore T does not attain its minimum. The general case follows by using the countable total family directly and choosing unit vectors in successive intersections of its kernels which are separated from zero by one fixed functional. ∎

The hypotheses in Theorem 2.1 are sharp. If Y is finite-dimensional, every operator from an infinite-dimensional X into Y has a non-trivial kernel. Also, if X has no countable total set of functionals, every operator from X into ℓ_2 is non-injective; thus (X, ℓ_2) has the weak minimizing property even when X is non-reflexive.

Theorem 2.2. *Let X be reflexive. If Y does not contain an isomorphic copy of X, then (X, Y) has the weak minimizing property.*

Proof. Let $(x_n) \subset S_X$ be a non-weakly null minimizing sequence for $T \in L(X, Y)$. Since T is not an isomorphic embedding, $m(T) = 0$, and hence $Tx_n \longrightarrow 0$. By reflexivity, a subsequence converges weakly to some non-zero $x \in X$. Therefore $Tx = 0$, and T attains its minimum at $x/\|x\|$. ∎

In particular, (X, ℓ_1) and (X, c_0) have the weak minimizing property for every infinite-dimensional reflexive space X. The complementary embedding case is unstable under renorming.

Theorem 2.3. *Let X be reflexive and suppose that Y contains an isomorphic copy of X. Then Y admits an equivalent norm for which (X, Y) does not have the weak minimizing property.*

To indicate the construction, take an isomorphic embedding $T : X \longrightarrow Y$ and a normalized basic sequence (e_n) in X. Suitable numbers $t_n > 0$ are chosen so that $\|e_1/2 + t_n e_n\| = 1 - 1/(2n)$. On $T(X)$, take as a new unit ball the closed convex hull of

$$T(B_X/2) \ \cup \ \{ \pm T(e_1/2 + t_n e_n) : n \geq 2 \}.$$

This defines an equivalent norm, extendable to Y, for which the embedding has minimum one, does not attain it, and admits a minimizing sequence converging weakly to a non-zero multiple of e_1. Thus Theorems 2.1–2.3 give a reflexivity criterion and a near-complete isomorphic description of the weak minimizing property.

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THE RIGHT TOPOLOGY ON SUBSPACES

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Abstract

We address the problem whether or not the Right topology on Banach spaces respects subspaces. More precisely, is the Right topology on a closed subspace F of a Banach space E the Right topology of E restricted to F? After some partial solutions and counterexamples, we disclose an unexpected connection of this problem with the Bourgain-Delbaen space.

1 Introduction

Let E be a Banach space. The Mackey topology on the bidual E^{**} of E is the topology of uniform convergence on absolute convex weakly compact subsets of E^*. The Right topology $\tau_R(E)$ on E was introduced in [2] as follows: Considering E as a subspace of its bidual E^{**}, $\tau_R(E)$ is the topology induced on E by the Mackey topology on E^{**}. In this work we shall work with the characterization of the Right topology we describe next. Given a Banach space F and a weakly compact operator $T\colon E \to F$, the map

$$p_T\colon E \to \mathbb{R}\ ,\ p_T(x) = \|T(x)\|,$$

is a seminorm on E. By $\mathcal{W}(E,F)$ we denote the space of weakly compact operators from E to F. In [3] it was proved that the Right topology $\tau_R(E)$ is the locally convex topology on E generated by the family of seminorms

$$\{p_T : F \text{ is a Banach space and } T \in \mathcal{W}(E,F)\},$$

in the sense of [1, Theorem 8.3.3]. This means, among other things, that, if $(x_\lambda)_\lambda$ is a net on E, then

$$x_\lambda \overset{\tau_R(E)}{\longrightarrow} x \in E \iff T(x_\lambda) \longrightarrow T(x) \text{ in } F,$$

for every Banach space F and every operator $T \in \mathcal{W}(E,F)$.

A lot of research has been made about the Right topology, in this work we address the issue of whether or not the Right topology is inherited by subspaces. More precisely, given a closed subspace F of the Banach space E, is it true that the Right topology on F is the topology on F induced by the Right topology on E? We prove that the answer is negative in general, and we identify some special situations in which the answer is positive. An unexpected relationship with the Bourgain-Delbaen space is disclosed.

2 Main Results

If (X,τ) is a topological space and Y is a subset of X, then we denote by $\tau|_Y$ the topology on Y induced by τ (subspace topology). Our problem is the following: Does the equality

$$\tau_R(F) = \tau_R(E)|_F$$

hold for every Banach space E and every closed subspace F of E? Bearing in mind that the Right topology lies between the weak and norm topologies, and that these two topologies respect subspaces, our problem is a quite natural one. First we prove that one inclusion holds in general.

Proposition 2.1. *For every Banach space E and every closed subspace F of E, $\tau_R(E)|_F \subseteq \tau_R(F)$.*

Next we establish that, contrary to the cases of the weak topology and the norm topology, the desired equality does not hold in general for the Right topology.

Proposition 2.2. *If Rad denotes the closed subspace of $L_1[0,1]$ generated by the Rademacher functions, then $\tau_R(L_1[0,1])|_{Rad} \neq \tau_R(Rad)$.*

It is well known that the Right topology on a reflexive space coincides with the norm topology. Since closed subspaces of reflexive spaces are reflexive as well, the desired equality holds trivially for closed subspaces of a reflexive space. So, henceforth we focus on non-reflexive spaces. The desired equality holds in certain situations:

Proposition 2.3. (a) *If F is a complemented subspace of the Banach space E, then $\tau_R(F) = \tau_R(E)|_F$.*
(b) *If ℓ_∞ is a closed subspace of a Banach space E, then $\tau_R(\ell_\infty) = \tau_R(E)|_{\ell_\infty}$.*
(c) *If c_0 is a closed subspace of a separable Banach space E, then $\tau_R(c_0) = \tau_R(E)|_{c_0}$.*

To settle the reader's curiosity, we mention that the only Banach spaces on which every closed subspace is complemented are the ones that are isomorphic to a Hilbert space, which are all reflexive, hence uninteresting for us.

It is never easy to prove that a certain subspace of a Banach space fails to be complemented. Combining the two results above we recover the following well-known non-complementability:

Corollary 2.1. *Rad is not complemented in $L_1[0,1]$.*

We finish with the following obvious question: Are there non-reflexive Banach spaces E for which $\tau_R(F) = \tau_R(E)|_F$ for every closed subspace F of E? In [2], Bourgain (Fields Medal 1994) and Delbaen constructed an infinite dimensional non-reflexive Banach space BD which solved some important open problems in Banach space theory, in particular, the existence (or not) of an infinite dimensional $\mathcal{L}_\infty$-space with the Schur property. Surprisingly, the space BD also fits our purpose:

Theorem 2.1. *$\tau_R(F) = \tau_R(BD)|_F$ for every closed subspace F of the Bourgain-Delbaen space BD.*

It is worth mentioning that BD is the only non-reflexive space with the property above we are aware of.

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ORTHOGONAL PROJECTIONS ONTO INVARIANT SUBSPACES OF THE HARDY SPACE OVER THE BIDISC

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Abstract

The Invariant Subspace Problem (ISP) remains unsolved for infinite-dimensional separable Hilbert spaces and asks if every bounded operator has a non-trivial closed invariant subspace. Due to the existence of universal operators (in the sense of Rota [3]), the ISP can be solved by proving that every invariant subspace of a universal operator is non-minimal (see [1]). In these notes we study the behavior of restricted orthogonal projections to invariant subspaces of universal Toeplitz operators on the Hardy space over the bidisc.

1 Introduction

Let $L^2(\mathbb{T}^2)$ denote the usual Lebesgue space and $L^\infty(\mathbb{T}^2)$ the essentially bounded functions with respect to σ. The Hardy space $H^2(\mathbb{D}^2)$ is the Hilbert space of holomorphic functions f on $\mathbb{D}^2$ for which

$$\|f\|^2 := \sup_{0<r<1} \int_{\mathbb{T}^2} |f(r\zeta)|^2 d\sigma(\zeta) < \infty.$$

Let P denote the orthogonal projection from $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$. For a function $\varphi \in L^\infty(\mathbb{T}^2)$, the Toeplitz operator T_φ with symbol φ is defined by

$$T_\varphi f = P(\varphi f)$$

for $f \in H^2(\mathbb{D}^2)$. Under analogous conditions, the 1-dimensional Toeplitz operator with symbol φ will be denoted by t_φ.

Let $H^2(z)$ and $H^2(w)$ denote the classical Hardy spaces over $\mathbb{D}$ in the variables z and w respectively. Then $H^2(\mathbb{D}^2)$ may be defined as the $H^2(z)$-valued Hardy space

$$H^2(\mathbb{D}^2) = \left\{ g(z,w) = \sum_{n=0}^{\infty} g_n(z) w^n : \sum_{n=0}^{\infty} \|g_n\|^2_{H^2(z)} < \infty \right\}.$$

Thus, considering $H_n = H^2(z)w^n$ for each $n \in \mathbb{N}$, we have

$$H^2(\mathbb{D}^2) = \bigoplus_{n=0}^{\infty} H_n.$$

For each $n \in \mathbb{N}$, we denote by P_n the orthogonal projection of $H^2(\mathbb{D}^2)$ onto H_n.

Consider the following classes of invariant subspaces

$$\mathcal{M} = \{M \subset H^2(\mathbb{D}^2) : M \text{ is a minimal invariant subspace for } T_z^*\}$$

and

$$\mathcal{M}_0 = \{M \in \mathcal{M} : \dim(M) = 1\}.$$

Since T_z^* is an universal operator for $H^2(\mathbb{D}^2)$ if we have $\mathcal{M} \subset \mathcal{M}_0$, then the ISP is true. Note that to show that $\mathcal{M} \subset \mathcal{M}_0$ it is enough to show that if M is an T_z^*-invariant subspace such that $\dim(M) \neq 1$ then M is non minimal. The next result suggests that, under certain conditions for the projections P_n, the invariant subspaces of T_z^* are either minimal of dimension 1 or are non-minimal.

Proposition 1.1. *Let $M \subset H^2(\mathbb{D}^2)$ an invariant subspace of T_z^*. If there are $n, m \in \mathbb{N}$ such that $\overline{P_n(M)} \neq \overline{P_m(M)}$ where both are non-trivial, then either $M \in \mathcal{M}_0$ or $M \notin \mathcal{M}$.*

Considering that the ranges of projection by M play an important role in the structure of the minimality of M (see [2]), it is necessary to better understand their behavior. The next corollary helps us with this task.

Corollary 1.1. *If $M \in \mathcal{M}$ and $\dim(M) = \infty$, then there exists $V \subset H^2(\mathbb{D})$ such that $\dim(V) = \infty$. Moreover for all $n \in \mathbb{N}$, either $P_n(M) = \{0\}$ or $P_n(M)$ dense in V.*

2 Main Results

For each non-zero $g \in H^2(w)$, let $P_g : H^2(\mathbb{D}^2) \to H^2(z)$ be the operator defined by

$$P_g(f)(z) = \int_{\mathbb{T}} f(z, w)\overline{g}(w)dw = \langle f, g\rangle_w(z)$$

for $f \in H^2(\mathbb{D}^2)$, where the subscript under the inner product denotes the variable of integration. The following lemma relates the operator P_g with the projections P_n.

Lemma 2.1. *Let $g(w) = \sum_{i=0}^{\infty} a_i w^i \in H^2(w)$. For all $f(z, w) \in H^2(\mathbb{D}^2)$ we have*

$$P_g(f)(z) = \sum_{i=0}^{\infty} \overline{a_i} P_i(f)(z).$$

Lemma 2.2. *Let $g, h \in H^2(w)$ with $\langle g, h\rangle = 0$ and $M \subset H^2(\mathbb{D}^2)$ an invariant subspace of T_z^*. If both $P_g(M)$ and $P_h(M)$ are nonzero and $\overline{P_g(M)} \neq \overline{P_h(M)}$, then either $dim(M) = 1$ or $M \notin \mathcal{M}$.*

Corollary 2.1. *Let $f, g, h \in H^2(w)$ with $g \perp h$ and $f \perp h$. If $M \in \mathcal{M}$ and $P_f(M), P_g(M)$ and $P_h(M)$ are nonzero, then $\overline{P_f(M)} = \overline{P_g(M)} = \overline{P_h(M)}$.*

Theorem 2.1. *If $M \in \mathcal{M}$, then for each $g \in H^2(w)$ there is an invariant subspace $V \subset H^2(\mathbb{D})$ for t_z^* such that either $P_g(M) = \{0\}$ or $\overline{P_g(M)} = V$.*

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SHADOWING FOR WEIGHTED COMPOSITION OPERATORS ON SPACES OF CONTINUOUS FUNCTIONS

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Abstract

We characterize the shadowing property for weighted composition operators acting on a class of Banach spaces of continuous functions, which we call admissible spaces. As an application, we obtain complete characterizations of shadowing for bilateral weighted translations on the spaces $C_0(\mathbb{R})$ and $C_b(\mathbb{R})$.

1 Preliminaries

We start our preliminaries with the main dynamical concept of this work.

Definition 1.1. (Shadowing) Let X be a metric space with metric d and let $f : X \to X$ be a homeomorphism. Given $\delta > 0$, a *δ-pseudotrajectory* of f is a sequence $(x_j)_{i<j<k}$ in X, where $-\infty \leq i < k \leq \infty$ and $k - i \geq 3$, such that $d(f(x_j), x_{j+1}) \leq \delta$ for all $i < j < k - 1$. We say that f has the *shadowing property* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudotrajectory $(x_j)_{j\in\mathbb{Z}}$ of f is *ε-shadowed* by a real trajectory of f, that is, there exists $x \in X$ such that $d(x_j, f^j(x)) < \varepsilon$ for all $j \in \mathbb{Z}$.

Let X be a space of functions. A *weighted composition operator* is an operator of the form $C_{w,f}(\varphi) = (\varphi \circ f)w$ for every $\varphi \in X$, where f (called the *composition function*) and w (called the *weight function*) are suitably chosen functions. The following definitions describe the spaces on which the weighted composition operators will act.

Definition 1.2. (Invertible system) We say that the triple (Ω, f, w) is an *invertible system* if Ω is a locally compact Hausdorff space, $w : \Omega \to \mathbb{K}$ is a continuous function such that $0 < \inf_{x\in\Omega} |w(x)| \leq \sup_{x\in\Omega} |w(x)| < \infty$, and $f : \Omega \to \Omega$ is a homeomorphism.

Fix an invertible system (Ω, f, w). To simplify notation, we define the sequence of functions $(w^{[n]})_{n\in\mathbb{Z}}$, where $w^{[0]} := 1, w^{[1]} := w, w^{[-1]} := \frac{1}{w\circ f^{-1}}$, $w^{[n]} := (w \circ f^{n-1})w^{[n-1]}$, for $n \geq 2$, and $w^{[-n]} := \frac{1}{w\circ f^{-n}} w^{[-(n-1)]}$ for $n \geq 2$.

Let $X \subset \ell^\infty(\mathbb{Z})$. Consider the following condition:

(CB) The canonical vectors $e_n := (\delta_{n,j})_{j\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}}$ belong to X and X can be written as the topological direct sum $X = M \oplus N$, where

$$M = \{(x_n)_{n\in\mathbb{Z}} \in X : x_n = 0, \forall\, n > 0\} \quad \text{and} \quad N = \{(x_n)_{n\in\mathbb{Z}} \in X : x_n = 0, \forall\, n \leq 0\}.$$

Definition 1.3. (Admissible space) Let (Ω, f, w) be an invertible system, and let $W \subset \Omega$ be a relatively compact and connected set with non-empty interior such that $W \subset \overline{\text{int}(W)}$. Let X be a closed subspace of $C_b(\Omega)$. We say that X is an *admissible space for $C_{w,f}$, generated by W*, if the following conditions hold:

(AS1) The weighted composition operator $C_{w,f} : X \to X$ is an invertible operator;

(AS2) $\Omega = \bigsqcup_{k\in\mathbb{Z}} f^k(W)$;

(AS3) There exist continuous functions $\rho_-, \rho_+ : \Omega \to \mathbb{K}$ such that $\rho_\pm \varphi \in X$ for all $\varphi \in X$, $\|\rho_\pm\|_\infty = 1$, $\operatorname{supp}(\rho_-) = \Omega_-^0$, $\operatorname{supp}(\rho_+) = \Omega_+^0$ and $\rho_- + \rho_+ \equiv 1$, where

$$\Omega_+^n := \overline{\bigsqcup_{k=n}^{\infty} f^k(W)} \quad \text{and} \quad \Omega_-^n := \overline{\bigsqcup_{k=n}^{\infty} f^{-k}(W)} \quad \text{for all } n \in \mathbb{N}_0;$$

(AS4) For each $x \in \Omega$, the set $S_x := \{(\varphi(f^n(x)))_{n\in\mathbb{Z}} : \varphi \in X\}$ is closed in $\ell^\infty(\mathbb{Z})$ and satisfies condition (CB);

(AS5) $C_c(\Omega) \subset X$.

2 Main results

In the following, we present the main result of [1], which provides a complete characterization of the shadowing property for weighted composition operators acting on admissible spaces.

Theorem 2.1. *[1, Theorem 32] Let (Ω, f, w) be an invertible system, where Ω is a metric space, w is uniformly continuous, and f is distance-preserving. Let X be an admissible space for $C_{w,f}$, generated by W. Then the weighted composition operator $C_{w,f}$ on X has the shadowing property if and only if one of the following conditions holds:*

(A) $\lim_{n\to\infty} \left\|w^{[n]}\right\|_\infty^{\frac{1}{n}} < 1$;

(B) $\lim_{n\to\infty} \left\|w^{[-n]}\right\|_\infty^{\frac{1}{n}} < 1$;

(C) $\lim_{n\to\infty} \left\|w^{[n]}\right\|_{\infty,\,\Omega_-^n}^{\frac{1}{n}} < 1$ *and* $\lim_{n\to\infty} \left\|w^{[-n]}\right\|_{\infty,\,\Omega_+^n}^{\frac{1}{n}} < 1$.

By specializing Theorem 2.1 to the setting of weighted translations, we obtain the following corollary:

Corollary 2.1. *[1, Corollary 34] Consider $X := C_0(\mathbb{R})$ or $X := C_b(\mathbb{R})$. Let $w : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function that satisfies $0 < \inf_{x\in\mathbb{R}} |w(x)| \le \sup_{x\in\mathbb{R}} |w(x)| < \infty$. Then the bilateral weighted translation operator $T_w : \varphi \in X \mapsto \varphi(\cdot + 1)w(\cdot) \in X$ has the shadowing property if and only if one of the following conditions holds:*

(a) $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |w(x)\cdots w(x+n-1)|^{\frac{1}{n}} < 1$;

(b) $\lim_{n\to\infty} \inf_{x\in\mathbb{R}} |w(x)\cdots w(x+n-1)|^{\frac{1}{n}} > 1$;

(c) $\lim_{n\to\infty} \sup_{x\in(-\infty,-n+1]} |w(x)\cdots w(x+n-1)|^{\frac{1}{n}} < 1$ *and*
$\lim_{n\to\infty} \inf_{x\in[n,\infty)} |w(x-n)\cdots w(x-1)|^{\frac{1}{n}} > 1$.

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XIX ENAMA - Novembro 2026 *23–24*

TOPOLOGIES DEFINED BY OPERATOR CLASSES ON BANACH SPACES

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Abstract

We study locally convex topologies generated by classes of linear operators on a fixed Banach space. For an operator class $C(E)$, we define the topology $\tau_{C(E)}$ induced by the seminorms $x \mapsto \|T(x)\|$, where $T \in C(E)$. We compare this topology with the weak and norm topologies, characterize continuous operators, and investigate the injective hull of an operator class.

1 Introduction

Let E be a Banach space. Given an operator class $C(E)$, the seminorms $x \mapsto \|T(x)\|$, where $T \in C(E)$, generate a natural locally convex topology $\tau_{C(E)}$ on E. This construction extends several well-known topologies, including the Right topology generated by weakly compact operators.

The aim of this paper is to study the topology $\tau_{C(E)}$, its relationship with the weak and norm topologies, the corresponding continuous operators, and the injective hull of an operator class. Our approach applies to important classes such as finite-rank, approximable, compact, weakly compact, completely continuous, and quasi Dunford–Pettis operators.

2 Main results

The construction of the topology $\tau_{\mathcal{C}(E)}$ is motivated by the idea of extending the Right topology to more general classes of operators. Instead of restricting ourselves to weakly compact operators, we consider arbitrary classes satisfying natural stability properties.

Definition 2.1. An *operator class* on the Banach space E, denoted by $C(E)$, is a correspondence that assigns to every Banach space F a subset $C(E,F)$ of $\mathcal{L}(E,F)$ satisfying:
(a) $C(E,F)$ is a vector subspace of $\mathcal{L}(E,F)$.
(b) $C(E,\mathbb{K}) = E^*$.
(c) If $U \in C(E,F)$ and $T \in \mathcal{L}(F,G)$, then $T \circ U \in C(E,G)$.

Important examples are the classes $\mathcal{F}(E)$ of finite-rank operators, of approximable operators, of compact operators, of weakly compact operators, and of completely continuous operators.

From now on, $C(E)$ is an operator class on the Banach space E. The following theorem shows that the two natural constructions of the topology, one by families of operators and the other by seminorms, actually define the same locally convex topology.

Theorem 2.1. *The topology on E generated by the family of functions $\{T \in C(E,F) : F$ is a Banach space$\}$ coincides with the locally convex topology on E determined by the family of seminorms*

$$\mathcal{R} = \{x \in E \mapsto \|T(x)\| : F \text{ is a Banach space and } T \in C(E,F)\}.$$

The topology above shall be denoted by $\tau_{C(E)}$. The first step is to compare $\tau_{C(E)}$ with the weak and norm topologies. This allows us to place $\tau_{\mathcal{C}(E)}$ among the classical locally convex topologies and motivates the characterization results below.

Theorem 2.2. (a) $\tau_w \subseteq \tau_{C(E)} \subseteq \tau_{\|\cdot\|}$.
(b) $\tau_{C(E)} = \tau_w$ *if and only if* $C(E) = \mathcal{F}(E)$.
(c) *If the class* $C(E)$ *is closed, then* $\tau_{C(E)} = \tau_w$ *if and only if* E *is finite dimensional.*

Theorem 2.3. *Let* $C(E)$ *and* $D(F)$ *be operator classes on the Banach spaces* E *and* F*, respectively, and let* $T : E \to F$ *be linear. Consider:*
(a) T *is norm continuous.*
(b) T *is* $\tau_{C(E)}$–$\tau_{D(F)}$ *continuous.*

Then (b) *implies* (a)*. If the two classes come from the same operator ideal, that is, if there is an operator ideal* I *such that* $C(E) = I(E)$ *and* $D(F) = I(F)$*, then (a) and (b) are equivalent.*

Following the classical construction of the injective hull of an operator ideal, we consider the corresponding injective hull of an operator class, denoted by $C^{\text{inj}}(E)$. This provides a natural extension of the class while preserving its essential properties.

Theorem 2.4. *The following are equivalent for an operator class* $C(E)$ *on a Banach space* E*:*
(a) $C(E)$ *is injective.*
(b) $C(E) = C^{\text{inj}}(E)$.
(c) $C(E)$ *has the domination property.*

Theorem 2.5. *Let* $C(E)$ *be an operator class on the Banach space* E*, let* F *be a Banach space, and let* $T \in \mathcal{L}(E, F)$*. Consider:*
(a) $T \in C(E, F)$.
(b) $T : (E, \tau_{C(E)}) \to F$ *is continuous.*
(c) $T \in C^{\text{inj}}(E, F)$.

Then (a) $\Rightarrow$ (b) $\Longleftrightarrow$ (c)*. The three conditions are equivalent whenever* $C(E)$ *is injective.*

Proposition 2.1. *Let* $C_1(E)$ *and* $C_2(E)$ *be two operator classes on the Banach space* E*. Then* $\tau_{C_1(E)} \subseteq \tau_{C_2(E)}$ *if and only if* $C_1(E) \subseteq C_2^{\text{inj}}(E)$.

Theorem 2.6. *Let* $C(E)$ *be an operator class on the Banach space* E*. Then* $\tau_{C(E)} = \tau_{\|\cdot\|}$ *if and only if* $C^{\text{inj}}(E) = \mathcal{L}(E)$.

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THREE DIFFERENT PROOFS THAT THE RATIOS BETWEEN CONSECUTIVE FIBONACCI NUMBERS CONVERGE TO THE GOLDEN NUMBER

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Abstract

This article presents three different proofs for the interesting fact that the ratios between consecutive Fibonacci numbers converge to the Golden Number. In more precise terms, it is shown, in three different ways, that the sequence $(x_n)_n$ defined by $x_n = f_{n+1}/f_n$, where f_n is the nth term of the Fibonacci Sequence, converges to $\varphi = \frac{1+\sqrt{5}}{2}$.

1 Introduction

Part of this article was published in *Revista do Professor de Matemática* No. 105, in which we presented only one of the three proofs that will be discussed here — that which uses Binet's Formula. To that version, we have added two other demonstrations grounded in Real Analysis results: one that establishes that the sequence of ratios is Cauchy and another that shows that the subsequences of even and odd indices are monotonic and bounded.

The Fibonacci Sequence has its origin in the famous Rabbit Problem proposed by Leonardo Fibonacci in *Liber Abaci* (13th century). Denoting by $(f_n)_n$ the Fibonacci Sequence, we have $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. The Golden Number φ arises from the division of a segment in extreme and mean ratio, being the positive root of $x^2 - x - 1 = 0$, given by:

$$\varphi = \frac{1+\sqrt{5}}{2}. \tag{1}$$

It is widely known that $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi$. However, most texts start from the undemonstrated assumption that the sequence is convergent. This article aims to present three rigorous demonstrations of convergence, extending the results to generalized Fibonacci sequences and negative indices.

2 Main Results and Proofs

The nth Fibonacci number can be explicitly expressed via Binet's Formula:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{2}$$

Let $x_n = \dfrac{f_{n+1}}{f_n}$. We analyze three distinct proofs for $\lim\limits_{n\to\infty} x_n = \varphi$:

1. **First proof (Binet):** Substituting equation (2) into the ratio yields:

$$x_n = \frac{1+\sqrt{5}}{2} \cdot \frac{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n+1}}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n}}. \tag{3}$$

Since the absolute value of the base is less than 1, the fractional terms vanish as $n \to \infty$, implying $x_n \to \varphi$.

2. **Second proof (Monotonic Subsequences):** From the recurrence relation $x_{n+1} = 1 + \frac{1}{x_n}$, one shows that (x_{2n}) is decreasing and bounded below, while (x_{2n-1}) is increasing and bounded above. Both converge to φ.

3. **Third proof (Cauchy Sequence):** Using the properties of Fibonacci numbers, one proves that $|x_{n+1} - x_n| \leq \left(\frac{1}{2}\right)^{n-2}$. By the triangle inequality and geometric series summation, (x_n) is Cauchy and hence convergent to φ.

Furthermore, for a Generalized Fibonacci Sequence $(F_n)_n$ where $F_1 = a$, $F_2 = b$, and $F_{n+1} = F_n + F_{n-1}$, we have $\lim F_{n+1}/F_n = \varphi$ whenever $a + b\varphi \neq 0$. Extending to negative indices gives $f_n = (-1)^{n+1} f_{-n}$ and limit $-\frac{1}{\varphi}$.

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ROBUST STRUCTURES AND CYCLIC OPERATORS

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Abstract

We investigate the structural and dynamical properties of linear operators on infinite-dimensional FrÃ©chet spaces. We first establish the existence of infinite-dimensional subspaces within the family of non-cyclic operators in $\mathcal{L}(X)$ that are both dense and closed with respect to the strong operator topology. Among other results, we show specifically that non-cyclic operators on ℓ_p (for $0 < p < \infty$) contain an isometric copy of ℓ_p.

1 Introduction

Although our operators will mainly act in a Banach or Fréchet space, the concepts to be handled make sense in more general settings. Let X be a (Hausdorff) topological vector space over the field $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{L}(X)$ denote the vector space of all (linear, continuous) operators from X into itself. We consider the following three dynamical properties of the members of $\mathcal{L}(X)$. If $x_0 \in X$ and $T \in \mathcal{L}(X)$, then the orbit and the projective orbit of x_0 under T are defined respectively as the sets

$$\mathcal{O}(T; x_0) := \{T^n x_0 : n \geq 0\} \quad \text{and} \quad \mathbb{K} \cdot \mathcal{O}(T; x_0) := \{\alpha T^n x_0 : n \geq 0, \alpha \in \mathbb{K}\}.$$

The operator T is *hypercyclic* (resp., *supercyclic, cyclic*), when there exists a vector $x_0 \in X$ such that $\mathcal{O}(T; x_0)$ (resp., $\mathbb{K} \cdot \mathcal{O}(T; x_0)$, $\text{span}(\mathcal{O}(T; x_0))$) is dense in X, and x_0 is said a hypercyclic (resp., supercyclic, cyclic) vector for T. It is plain that hypercyclicity implies supercyclicity and, in turn, supercyclicity implies cyclicity. Moreover, it is evident that if X supports a cyclic (so, a supercyclic, or a hypercyclic) operator, then X must be separable. While cyclicity is a long-standing notion in operator theory, hypercyclicity and supercyclicity are relatively new.

It is natural to investigate the lineability of the complementary family $\mathcal{L}(X) \setminus \mathcal{HC}(X)$, that is, the set of all bounded operators that fails to be hypercyclic. The answer is positive for locally convex spaces and, in a stronger way, for Banach spaces:

Proposition 1.1. (a) *For every infinite-dimensional locally convex space X, the set $\mathcal{L}(X) \setminus \mathcal{HC}(X)$ is lineable.*

(b) *For every infinite-dimensional Banach space X, the set $\mathcal{L}(X) \setminus \mathcal{HC}(X)$ is spaceable with respect to the operator norm topology.*

2 Main Results

It is of particular interest to study the linear size of the *complements* of the classes of cyclic, injective, and dense-range operators (denoted by $\mathcal{CYC}(X)$, $\mathcal{INJ}(X)$, and $\mathcal{DR}(X)$, respectively). In fact, we are able to simultaneously capture the absence of all three properties: cyclicity, dense range, and injectivity. Let us recall that natural topology on $\mathcal{L}(X)$ is the so-called *strong operator topology* (SOT). In it, a net $(T_\alpha)_{\alpha \in I}$ tends to T if and only if $T_\alpha(x) \to T(x)$ for every $x \in X$. In the case that X is a Banach space, it is a well-known fact that the SOT topology is weaker than the norm topology on $\mathcal{L}(X)$ (the norm is $\|T\| = \sup_{\|x\|=1} \|T(x)\|$), and that both topologies coincide if and only if X is finite dimensional.

Theorem 2.1. *Let X be an infinite-dimensional locally convex space. Then the set*

$$\mathcal{L}(X) \setminus (\mathcal{DR}(X) \cup \mathcal{CYC}(X) \cup \mathcal{IN}\mathcal{J}(X))$$

is SOT-dense-lineable in $\mathcal{L}(X)$.

Theorem 2.2. *Let X be an infinite-dimensional Fréchet space. Then the set*

$$\mathcal{L}(X) \setminus (\mathcal{DR}(X) \cup \mathcal{CYC}(X) \cup \mathcal{IN}\mathcal{J}(X))$$

is SOT-spaceable in $\mathcal{L}(X)$.

The following consequence of Theorem 2.1 is extracted from the inclusions and from the fact that, if X is a Banach space, then the SOT topology is weaker than the operator norm topology.

Corollary 2.1. *Let X be an infinite-dimensional Fréchet space. Then the sets $\mathcal{L}(X) \setminus \mathcal{DR}(X)$, $\mathcal{L}(X) \setminus \mathcal{CYC}(X)$, $\mathcal{L}(X) \setminus \mathcal{IN}\mathcal{J}(X)$, $\mathcal{L}(X) \setminus \mathcal{SRJ}(X)$, $\mathcal{L}(X) \setminus \mathcal{SC}(X)$ and $\mathcal{L}(X) \setminus \mathcal{HC}(X)$ are SOT-dense-lineable and SOT-spaceable in $\mathcal{L}(X)$. In particular, if X is an infinite-dimensional Banach space, then all of these sets are spaceable in $\mathcal{L}(X)$ with respect to the operator norm topology.*

It follows from the previous Corollary that for an infinite-dimensional Banach space X the family of non-cyclic operators is norm-spaceable in $\mathcal{L}(X)$. In the case that X is the sequence space ℓ_p $(1 \leq p < \infty)$, we can improve this result in the way given in the next theorem. In fact, we can also get that the operators under analysis do not have dense range. In addition, the result will also hold in the non-locally convex case $0 < p < 1$. In this case, we consider in $\mathcal{L}(\ell_p)$ the distance $d(T, S) = \|T - S\|$ generated by the mapping $\|T\| := \sup\{\|T(x)\|_p : \|x\|_p \leq 1\}$, where $\|(x_n)\|_p = \sum_{n=1}^{\infty} |x_n|^p$. It is easy to check that $\|\cdot\|$ is an F-norm generating the topology of uniform convergence on bounded subsets of ℓ_p.

Theorem 2.3. *For every $p > 0$, the set $\mathcal{IN}\mathcal{J}(\ell_p) \setminus (\mathcal{CYC}(\ell_p) \cup \mathcal{DR}(\ell_p))$ contains, except for zero, an isometric copy of ℓ_p.*

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THE WEAKLY P-SINGULAR MAXIMIZING PROPERTY

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Abstract

Motivated by recent developments concerning the weak maximizing property (WMP) for norm-attaining operators, this work investigates its natural p-analogues. We demonstrate that a direct translation via weak p-summability is insufficient and that more restrictive conditions on maximizing sequences are needed to yield a meaningful mathematical concept. To resolve this, we introduce a new type of maximizing property of order p. Within this framework, we establish a characterization of the Schur property of order p. Finally, we obtain structural characterizations of the Banach spaces enjoying this new maximizing property of order p.

1 Introduction

Let $1 < p < \infty$ and $1 \leq q < \infty$. In [1, Theorem 1], the authors give a characterization for the set of *norm attaining operators* $T : \ell_p \to \ell_q$, which states that T attains its norm if and only if there exists a non weakly null maximizing sequence for T. Recall that by a maximizing sequence, we mean a sequence $(x_n)_n$ such that $\lim_{n\to\infty} ||Tx_n|| = ||T||$ and $||x_n|| = 1$ for every $n \in \mathbb{N}$. This result has motivated the authors of [2] to introduce a new property related to the study of norm attaining operators, known as *the weak maximizing property* (WMP, in short). A pair of Banach spaces (X, Y) satisfies the WMP if, for every $T : X \to Y$, the existence of a non weakly null maximizing sequence for T implies that T attains its norm. Although this property seems primarily related to the topologies on the underlying spaces, making it largely an analytic property it is deeply intertwined with the geometry of the spaces.

Following a well-established line of research in Banach spaces, it seems natural to ask whether it is possible to define a p-version of the *weak maximizing property*. Replacing non-weak nullity with non-weak p-summability seems to be a natural path to do that; however, such an attempt does not define a meaningful property. To do so correctly, we must impose more restrictive conditions on the maximizing sequences.

Definition 1.1. *Let $1 \leq p < \infty$. A sequence $(x_n)_n$ in a Banach space X is called* ***weakly p-singular*** *if it does not admit weakly p-summable subsequences. Also, we say that the pair of Banach spaces (X, Y) satisfy the* ***weakly p-singular maximizing property*** *(SMP_p, for short) if for every continuous operator $T : X \to Y$ the existence of a weakly p-singular maximizing sequence for T implies that T attains its norm.*

2 Main Results

The main results are stated as follows:

Proposition 2.1. *Let X and Y be Banach spaces such that (X, Y) has the SMP_p for some $1 \leq p < \infty$.*
(1) *If $q > p$ then the pair (X, Y) has the SMP_q.*
(2) *The pair (X, Y) has the WMP.*

Theorem 2.1. *Let $1 \leq q, r < \infty$ and $1 < p < \infty$ be given.*
(1) *The pair (ℓ_p, ℓ_q) has the SMP_r if and only if $q < p$ or $r \geq p^*$.*

(2) *The pair* (ℓ_p, c_0) *has the* SMP_r *if and only if* $r \geq p^*$.
(3) *If* H *and* K *are Hilbert spaces, then* (H, K) *has the* SMP_r *for every* $r \geq 2$. *This condition is sharp since the pair* $(L_2([0,1]), L_2([0,2]))$ *does not have the* SMP_r *for* $1 \leq r < 2$.

In [3], a pair (X, Y) has the WMP for every reflexive space X if and only if Y has the Schur property. Extending this, we characterize the Schur property of order p (SP_p). Here, we replace domain reflexivity with *weak p-precompactness*, meaning every sequence $(x_n)_n \subset K$ admits a weakly p-convergent subsequence.

Theorem 2.2. *For a Banach space* Y *and* $1 < p < \infty$, *the following are equivalent:*
(1) Y *has the* SP_p.
(2) *For every Banach space* X *such that* B_X *is relatively weakly p-precompact, every bounded linear operator* $T : X \to Y$ *is norm-attaining.*
(3) *For every* $1 \leq r < \infty$ *and every Banach space* X *such that* B_X *is relatively weakly p-precompact, the pair* (X, Y) *has the* SMP_r.
(4) *For every Banach space* X *such that* B_X *is relatively weakly p-precompact, the pair* (X, Y) *has the WMP.*

Proposition 2.2. *Let* $1 \leq p < \infty$. *If* (X, Y) *has the* SMP_p, *and that* $K, T : X \to Y$ *are p-convergent, repectively, bounded linear operators. If* $||T|| < ||T + K||$ *then* $T + K$ *attains its norm.*

However, the converse of the above proposition is not true.

Theorem 2.3. *If* B_X *is a relatively weakly p-precompact set, then there exist a Banach space* Y *such that* B_Y *is relatively weakly p-precompact, a bounded linear operator* $T : X \to Y$, *and a rank one operator* $R : X \to Y$ *such that* $||T|| < ||T + R||$ *and* $T + R$ *fails to attains its norm.*

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *31–32*

CYCLICITY AND COMPLEX SYMMETRY OF COMPOSITION OPERATORS ON PALEY-WIENER SPACES

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Abstract

In this article, we characterize the cyclicity of bounded composition operators $C_\phi f = f \circ \phi$ on the Paley-Wiener spaces of entire functions PW_σ for $\sigma > 0$. We show that C_ϕ is cyclic precisely when $\phi(z) = z + b$ where either $b \in \mathbb{C} \setminus \mathbb{R}$ or $b \in \mathbb{R}$ with $0 < |b| \leq \pi/\sigma$. We also describe when the reproducing kernels of PW_σ are cyclic vectors for C_ϕ and see that this is related to a question of completeness of exponential sequences in $L^2[-\sigma, \sigma]$.

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space of complex-valued holomorphic functions on a set X. A composition operator C_ϕ on $\mathcal{H}$ with symbol $\phi : X \to X$ is defined as $C_\phi f = f \circ \phi$, for $f \in \mathcal{H}$. The cyclicity of composition operators has been studied on several holomorphic function spaces and has become an active research topic within linear dynamics.

We now introduce the main framework of our study: **composition operators on Paley-Wiener space**. For $\sigma > 0$, the Paley-Wiener space PW_σ consists of all entire functions of exponential type less than or equal to σ, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. Each space PW_σ is a Hilbert space when endowed with the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt.$$

The norm induced by this inner product makes PW_σ a reproducing kernel Hilbert space, this means that the point evaluations of functions on PW_σ are continuous linear functionals. Indeed, if $w \in \mathbb{C}$ and $f \in PW_\sigma$, then $f(w) = \langle f, k_w \rangle$ where the reproducing kernel k_w is given by

$$k_w(z) = \frac{\sin \sigma(z - \overline{w})}{\pi(z - \overline{w})}, \quad z \in \mathbb{C}.$$

In [1], Chacón, Chacón and Giménez initiated the study of composition operators on the classical Paley-Wiener space PW_π. They prove that C_ϕ is bounded on PW_π precisely when

$$\phi(z) = az + b, \text{ where } a \in \mathbb{R} \text{ with } 0 < |a| \leq 1 \text{ and } b \in \mathbb{C}. \tag{1}$$

More recently Ikeda, Ishikawa and Yoshihiro [3] show that this is true even in the general context of reproduction kernel Hilbert spaces of entire functions on $\mathbb{C}^n$.

In this work, we study the cyclicity and complex symmetry of composition operators on the Paley-Wiener spaces PW_σ. From results in [3] it follows that the only bounded composition operators C_ϕ on PW_σ are those induced by symbols of the form (1). The Paley-Wiener spaces PW_σ are isometrically embedded into Hardy-Hilbert space $H^2(\mathbb{C}_+)$ of the half-plane as so-called *model subspaces* K_Θ defined as

$$K_\Theta := H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+) \quad \text{where} \quad \Theta(z) = e^{i\sigma z}.$$

Therefore one can view this work as the study of C_ϕ on some model subspaces of $H^2(\mathbb{C}_+)$. In contrast with $H^2(\mathbb{C}_+)$ ([2]), our results show the existence of non-normal complex symmetric C_ϕ, and that the cyclicity of C_ϕ and its adjoint C_ϕ^* in PW_σ depends on $\sigma > 0$. In particular, we show that no C_ϕ is supercyclic on any PW_σ. Finally, we characterize the reproducing kernels in PW_σ that are cyclic vectors for C_ϕ and show that this is closely related to the completeness of exponential sequences $(e^{i\lambda_n t})_{n\in\mathbb{N}}$ in $L^2[-\sigma,\sigma]$.

2 Main Results

Some of the our main results (see [2]) are summarized in the following table.

Table 1: Main results for composition operators on each Paley-Wiener space B_σ^2.

C_ϕ cyclic	$a = 1$ with $b \in \mathbb{C} \setminus \mathbb{R}$ or $0 < \lvert b\rvert \leq \pi/\sigma$, $b \in \mathbb{R}$
C_ϕ^* cyclic	$0 < \lvert a\rvert < 1$, $b \in \mathbb{C}$ or C_ϕ cyclic
C_ϕ supercyclic	Never
C_ϕ complex symmetric	$a = 1$, $b \in \mathbb{C}$ or $a = -1$, $b \in \mathbb{C}$
C_ϕ normal	$a = 1$, $b \in \mathbb{C}$ or $a = -1$, $b \in \mathbb{R}$

In addition, we answer the following question: **Are any of the reproducing kernels $(k_w)_{w\in\mathbb{C}}$ cyclic vectors for C_ϕ?** The answer (Theorem 2.1) reveals an interesting dichotomy and a connection with the question of completeness of exponential sequences in $L^2[-\sigma,\sigma]$.

Theorem 2.1. *Let $\phi(z) = z + b$ where $b \in \mathbb{C}$. Then the kernels $(k_w)_{w\in\mathbb{C}}$ in PW_σ are:*

- *all cyclic vectors for C_ϕ when $b \in \mathbb{C} \setminus \mathbb{R}$ or $0 < \lvert b\rvert < \pi/\sigma$ and $b \in \mathbb{R}$;*
- *never cyclic for C_ϕ when $\lvert b\rvert = \pi/\sigma$.*

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *33–34*

GEOMETRIC ASPECTS OF THE LIPSCHITZ-FREE SPACES

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Abstract

This projects aims to study recent developments concerning Lipschitz-free spaces, with emphasis in the results due to Aliaga, García-Lirola, Guirao, Pernecká, Petitjean, Procházka, Rueda Zoca, Smith and Weaver, that classify the extreme points of the unit ball of the spaces $\mathcal{F}(M)$ and guarantee the existence of supports to the elements of $\mathcal{F}(M)$ [1, 2, 3, 4, 5, 7, 8, 9, 10].

1 Introduction

Let $(M, d, 0)$ be a pointed metric space. $\mathrm{Lip}_0(M)$ denotes the (real) Banach space of all the Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0) = 0$, with norm given by the Lipschitz constant,

$$L(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \ : \ x, y \in M, x \neq y \right\}. \tag{1}$$

Let $\delta_M : M \to \mathrm{Lip}_0(M)^*$ be the evaluation map. The *Lipschitz-free space* of M is defined as $\mathcal{F}(M) = \overline{\mathrm{span}}\ \delta_M(M)$. For each distinct points $x, y \in M$, consider the *molecule* $m_{xy} = \frac{\delta_M(x) - \delta_N(y)}{d(x,y)} \in S_{\mathcal{F}(M)}$.

The theoretical relevance of the Lipschitz-free spaces is due to the fact that, for every Banach space X and every Lipschitz function $f : M \to X$ such that $f(0) = 0$, there exists a unique bounded linear operator $T : \mathcal{F}(M) \to X$ such that $T \circ \delta_M(x) = f(x)$ for every $x \in M$. Because of it, problems related to Lipschitz functions can be linearized, at the cost of making the spaces involved more complicated.

We will also make use of the concepts of extreme, exposed, w^*-extreme, denting and strongly exposed points, whose definitions are available at [6].

The problem of the characterization of the extreme points of the unit ball of $\mathcal{F}(M)$ was chosen to guide the work. It was open since 1999 (when [10] was first published) and it was recently completely solved in [4], after many partial results [1, 2, 3, 5, 7, 8]. The solution makes use of tools involving Banach algebras and lattices, compactifications and Choquet theory.

2 Main goals

For $x, y \in M$, let $I(x, y) = \{z \in M : d(x, z) + d(z, y) = d(x, y)\}$. It is clear that if there exists $z \in I(x, y) \setminus \{x, y\}$, then $m_{xy} = t m_{xz} + (1 - t) m_{zy}$ for $t = \dfrac{d(x, z)}{d(x, y)} \in]0, 1[$, so m_{xy} is not an extreme point of $B_{\mathcal{F}(M)}$. Meanwhile, in [10] it is proved that every w^*-extreme point of $B_{\mathcal{F}(M)}$ must be a molecule.

The main results the project aims to cover are:

Theorem 2.1 ([7], Theorem 5.4)**.** *The strongly exposed points of $B_{\mathcal{F}(M)}$ are precisely the molecules m_{xy} such that*

$$\exists c > 0 : \forall z \in M \setminus \{x, y\}, \min\{d(x, z), d(z, y)\} \leq c \cdot (d(x, z) + d(z, y) - d(x, y)). \tag{2}$$

Theorem 2.2 ([1], Theorem 4.1)**.** *The w^*-extreme points of $B_{\mathcal{F}(M)}$ are precisely the molecules m_{xy} such that*

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall z \in M \setminus (B(x, \varepsilon) \cup B(y, \varepsilon)), d(x, z) + d(z, y) \geq d(x, y) + \delta. \tag{3}$$

Theorem 2.3 ([8], Theorem 2.4)**.** *Every w^*-extreme point of $B_{\mathcal{F}(M)}$ is a denting point.*

In [2], the authors define the *support* of an element $\mu \in \mathcal{F}(M)$ as the smallest closed subset K of M such that $\mu \in \mathcal{F}_M(K) = \overline{\text{span}}\ \delta_M(K) \subset \mathcal{F}(M)$. Note that $\mathcal{F}_M(K) \equiv \mathcal{F}(K \cup \{0\})$. They prove that every element of $\mathcal{F}(M)$ has a support, which is equivalent to the following theorem:

Theorem 2.4 (Intersection Theorem; [2], Theorem 3.3 and [3], Theorem 2.1)**.** *Let $\{K_i : i \in I\}$ be a family of closed subsets of M. Then*

$$\bigcap_{i\in I} \mathcal{F}_M(K_i) = \mathcal{F}_M\left(\bigcap_{i\in I} K_i\right). \tag{4}$$

Theorem 2.5 ([4], Theorem 3.1)**.** *The extreme points of $B_{\mathcal{F}(M)}$ are precisely the molecules m_{xy} such that $x, y \in M$ are distinct and $I(x, y) = \{x, y\}$.*

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XIX ENAMA - Novembro 2026 *35–36*

COPIES OF ℓ_∞ IN THE RELATIVE COMPLEMENTS OF THE SPACE OF HOLOMORPHIC LIPSCHITZ BI-FORMS

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Resumo

Let X and Y be nonzero complex Banach spaces. We study the space $\mathcal{H}BL_0(B_X, B_Y)$ of all locally bounded and separately holomorphic scalar-valued functions on $B_X \times B_Y$ which vanish on the coordinate axes and whose rectangular increments satisfy a mixed Lipschitz condition. We investigate the linear structure of the two relative complements arising from the identity $\mathcal{H}BL_0(B_X, B_Y) = \mathcal{H}_0^\infty(B_X, B_Y) \cap BLip_0(B_X, B_Y)$. We prove that the first relative complement, after adjoining the zero function, contains a closed linear subspace isomorphic to ℓ_∞. The second relative complement, after adjoining zero, contains a closed linear subspace isometrically isomorphic to ℓ_∞, with respect to the mixed Lipschitz norm. Consequently, both relative complements are ℓ_∞-spaceable, and in the second case the copy of ℓ_∞ can be chosen isometric.

1 Introduction

A natural problem in nonlinear functional analysis is to determine whether sets of functions failing a prescribed regularity condition contain large linear structures. This question belongs to the general theory of lineability and spaceability and, in the present work, is considered for scalar-valued functions defined on products of open unit balls. Our investigation is motivated by the corresponding one-variable theory of holomorphic Lipschitz functions developed in [2]. For background on holomorphic mappings on Banach spaces, we refer to [1], while related product-space constructions associated with bilinear mappings can be found in [3].

Throughout this work, X and Y are nonzero complex Banach spaces, and B_X and B_Y denote their open unit balls, respectively. For a function $F\colon B_X \times B_Y \to \mathbb{C}$, its rectangular increment is defined by

$$\Delta F(x_1, x_2; y_1, y_2) = F(x_1, y_1) - F(x_2, y_1) - F(x_1, y_2) + F(x_2, y_2).$$

We denote by $BLip_0(B_X, B_Y)$ the space of all functions $F\colon B_X \times B_Y \to \mathbb{C}$ satisfying $F(x, 0) = F(0, y) = 0$ for every $x \in B_X$ and $y \in B_Y$, and such that

$$\|F\|_{BL} := \sup_{\substack{x_1,x_2\in B_X,\ x_1\neq x_2\\ y_1,y_2\in B_Y,\ y_1\neq y_2}} \frac{|\Delta F(x_1, x_2; y_1, y_2)|}{\|x_1 - x_2\|\,\|y_1 - y_2\|} < \infty.$$

The space $\mathcal{H}BL_0(B_X, B_Y)$ consists of all locally bounded and separately holomorphic functions $F\colon B_X \times B_Y \to \mathbb{C}$ which vanish on the coordinate axes and satisfy $\|F\|_{BL} < \infty$. Here, separate holomorphy means that $F(\,\cdot\,, y)$ is holomorphic on B_X for every $y \in B_Y$, and $F(x, \,\cdot\,)$ is holomorphic on B_Y for every $x \in B_X$.

We also denote by $\mathcal{H}_0^\infty(B_X, B_Y)$ the space of all bounded and separately holomorphic functions on $B_X \times B_Y$ which vanish on the coordinate axes, endowed with the supremum norm

$$\|F\|_\infty = \sup_{(x,y)\in B_X\times B_Y} |F(x, y)|.$$

Since $\Delta F(x, 0; y, 0) = F(x, y)$, every $F \in BLip_0(B_X, B_Y)$ satisfies

$$|F(x, y)| \le \|F\|_{BL}\|x\|\|y\|, \qquad (x, y) \in B_X \times B_Y.$$

In particular, $\|F\|_\infty \leq \|F\|_{BL}$, and hence

$$\mathcal{H}BL_0(B_X, B_Y) = \mathcal{H}_0^\infty(B_X, B_Y) \cap BLip_0(B_X, B_Y).$$

Recall that a subset A of a topological vector space is said to be ℓ_∞-spaceable if $A \cup \{0\}$ contains a closed linear subspace isomorphic to ℓ_∞. Our main result shows that both relative complements determined by the preceding identity are ℓ_∞-spaceable. Moreover, for the relative complement inside $BLip_0(B_X, B_Y)$, the copy of ℓ_∞ can be chosen isometric.

2 Main Results

The identity established above gives rise to two natural relative complements. The first consists of bounded, separately holomorphic functions which fail the mixed Lipschitz condition, whereas the second consists of pointed mixed-Lipschitz functions which are not separately holomorphic. Although neither of these set differences is, in general, a vector space, both contain large closed linear structures after the zero function is adjoined. The conclusion for the second relative complement is stronger, since the copy of ℓ_∞ can be chosen isometric.

We endow $\mathcal{H}_0^\infty(B_X, B_Y)$ with the supremum norm and $BLip_0(B_X, B_Y)$ with the norm $\|\cdot\|_{BL}$. Our main result is the following.

Theorem 2.1. *Let X and Y be nonzero complex Banach spaces. Then the following assertions hold:*

(a) *The set*

$$\big(\mathcal{H}_0^\infty(B_X, B_Y) \setminus \mathcal{H}BL_0(B_X, B_Y)\big) \cup \{0\}$$

contains a closed linear subspace isomorphic to ℓ_∞.

(b) *The set*

$$\big(BLip_0(B_X, B_Y) \setminus \mathcal{H}BL_0(B_X, B_Y)\big) \cup \{0\}$$

contains a closed linear subspace isometrically isomorphic to ℓ_∞.

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XIX ENAMA - Novembro 2026 *37–38*

ON PROPERTIES OF THE PISIER-XU SPACE

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Abstract

We study some structural properties of the space defined by Pisier and Xu, and prove that it is primary.

1 Introduction

The classical James space, denoted by J, is one of the fundamental examples in Banach space theory. It was introduced by R. C. James in his study of bases and reflexivity. The space J is non-reflexive and is isometric to its second dual [3]. More precisely, J is quasi-reflexive of order one: $\dim(J^{**}/J) = 1$.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Denote by $\mathcal{P}$ the family of all finite increasing sequences of positive integers of odd length, $p = (p_1, \ldots, p_{2m+1}), \quad p_1 < \cdots < p_{2m+1}$. For $x = (x_n) \in c_{00}$, define

$$\|x\|_J = \sup_{p\in\mathcal{P}} \left(|x_{p_{2m+1}}|^2 + \sum_{j=1}^{m} |x_{p_{2j-1}} - x_{p_{2j}}|^2 \right)^{1/2}.$$

The James space J is the completion of c_{00} with respect to this norm.

The James space J satisfies the following structural properties. It contains no subspace isomorphic to c_0 and no subspace isomorphic to ℓ_1. If (u_n) is a normalized block basic sequence of the canonical basis (e_n) whose successive supports are separated by at least one index, then (u_n) is equivalent to the unit vector basis of ℓ_2. Casazza, Lin and Lohman proved that J contains uncountably many mutually non-equivalent unconditional basic sequences [2].

Another important structural property is that J is a primary Banach space [1]. Recall that a Banach space X is called primary if, whenever $X = Y \oplus Z$ as a topological direct sum, then either Y or Z is isomorphic to X.

Motivated by certain similarities with the construction of the James space, we study the structural properties of the space introduced by Pisier and Xu [5] via real interpolation. For a scalar sequence $x = (x_n)_{n\geq 0}$ set

$$\|x\|_{v_1} = |x_0| + \sum_{n=1}^{\infty} |x_n - x_{n-1}|.$$

Let v_1 be the Banach space of sequences with finite v_1 norm. We write $v_1^0 = v_1 \cap c_0$. For a compatible Banach couple (A_0, A_1), its K-functional is

$$K(t, x; A_0, A_1) = \inf_{x=x_0+x_1} \big(\|x_0\|_{A_0} + t\|x_1\|_{A_1}\big),$$

for $0 < \theta < 1$ and $1 \leq q < \infty$, the corresponding real interpolation norm is

$$\|x\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^{\infty} [t^{-\theta} K(t, x)]^q \, \frac{dt}{t} \right)^{1/q}$$

Definition 1.1. *The Pisier–Xu space is* $\widehat{W}_2 = (v_1, \ell_\infty)_{1/2,2}$. *The canonical zero-limit model is* $W_2 = (v_1^0, c_0)_{1/2,2}$.

Theorem 1.1 ([5, 4])**.** *The following hold.*

1. *W_2 is quasi-reflexive of order one:* $\dim(W_2^{**}/W_2) = 1$.

2. $W_2 \simeq W_2^*$.

3. *For every* $1 < r < 2 < s < \infty$, W_2 *has type* r *and cotype* s.

4. *There is* $C > 0$ *such that every* n*-dimensional subspace* $E \subset W_2$, $n > 1$, *satisfies* $d(E, \ell_2^n) \leq C \log n$.

2 Main Results

Following Casazza's proof for the James space [1] proof, we obtain

Theorem 2.1. *W_2 is primary.*

The proof relies on the following propositions.

Proposition 2.1. *Suppose $W_2 = R \oplus Y$, where R is reflexive and Y is quasi-reflexive of order one. Then Y contains a subspace $V \simeq W_2$ which is complemented in Y. Moreover, $Y = V \oplus R_0$ for a reflexive space R_0 complemented in W_2.*

Proposition 2.2. *If R is a complemented reflexive subspace of W_2, then $W_2 \oplus R \simeq W_2$.*

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XIX ENAMA - Novembro 2026 *39–40*

MATHEMATICAL MODELING OF PEPTIDE FORMATION UNDER PREBIOTIC CONDITIONS

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Abstract

The spontaneous formation of peptides is considered one of the fundamental steps in prebiotic chemical evolution. In this work, a probabilistic mathematical model combining experimental amino acid abundances and peptide bond formation yields was developed to investigate peptide formation. The model was implemented in GNU Octave to generate all peptide sequences composed of five amino acids and to estimate their relative formation probabilities. To quantify the selectivity of the process, a relevance ratio was defined. Numerical simulations demonstrate that only a very small subset of all possible peptides concentrates most of the total probability. The results indicate that peptide formation under prebiotic conditions may have been governed by highly selective mechanisms.

1 Introduction

Understanding the origin of life requires investigating the chemical processes responsible for the formation of biologically relevant molecules. Peptides are considered key compounds. Since the number of possible peptide sequences increases exponentially with chain length, exhaustive experimental investigation rapidly becomes impractical.

In this work, we propose a probabilistic mathematical model that combines experimental amino acid abundances with peptide bond formation yields to estimate peptide formation probabilities. The model also identifies a quantitative measure of the selectivity of peptide formation.

2 Mathematical Model and Results

The model considers five amino acids (Alanine, Glycine, Leucine, Valine and Proline), whose peptide bond formation yields were obtained from experimental studies (Bujdák and Rode, 1998; Bujdák and Rode, 2002). The probability of forming a dipeptide is

$$P(XY) = R_{XY}\,Ab(X)Ab(Y), \tag{1}$$

where R_{XY} is the experimental peptide bond yield and $Ab(\cdot)$ denotes the normalized amino acid abundances of the binding amino acids. For longer peptides, probabilities are computed recursively by incorporating the probability of the previously formed chain and the corresponding bond yield.

The total number of possible sequences of length n is $N(n) = 5^n$. After computing all probabilities, sequences are ranked in decreasing order. The minimum number M of peptides required to reach a cumulative probability threshold defines the relevance ratio

$$\rho = \frac{M}{5^n}. \tag{2}$$

Table 1 summarizes the results obtained for a cumulative probability threshold of 95%.

Table 1: Relevance ratio for a 95% cumulative probability threshold.

n	M	Total	ρ	Compute Time (s)
2	6	25	0.240000	13.72
3	11	125	0.088000	11.84
4	18	625	0.028800	13.24
5	30	3125	0.009600	8.54
6	49	15625	0.003136	10.80
7	80	78125	0.001024	12.39
8	129	390625	0.000330	28.60
9	204	1953125	0.000104	83.59
10	318	9765625	0.000033	472.57

3 Conclusions

Numerical simulations indicate that peptide formation is highly selective, with most of the total probability concentrated in a very small fraction of all possible sequences. The proposed relevance ratio provides an effective quantitative measure of this selectivity and may be extended to larger amino acid sets and alternative prebiotic scenarios.

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XIX ENAMA - Novembro 2026 *41–42*

CONSTRUCTING POSITIVITY-PRESERVING INTEGRATORS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

We present an unconditionally positivity-preserving numerical scheme for systems of Itô stochastic differential equations (SDEs) whose exact solutions remain almost surely positive. By construction, the proposed method guarantees strict positivity at each time step without restrictions on the step size, while maintaining a computational cost comparable to the standard explicit Euler–Maruyama scheme. Pathwise strong convergence of the method is established under suitable regularity assumptions on the drift and diffusion coefficients.

1 Introduction

Stochastic differential equations (SDEs) are widely used to model continuous-time dynamics in mathematical finance, biology, chemical kinetics, and population dynamics [3, 5]. In many of these applications, state variables represent physical quantities that are inherently strictly positive [4]. While the underlying continuous-time SDEs often preserve this positivity pathwise, standard explicit numerical discretizations generally fail to preserve the positivity constraint at the discrete level, frequently producing unphysical negative values. To address these limitations, several positivity-preserving discretizations have been developed in recent years (see e.g, [1, 2]). However, many of these schemes are often tailored to specific scalar problems or require restrictive global Lipschitz conditions that may not hold in general systems [4, 5].

In this work, we propose an explicit, unconditionally positivity-preserving integrator for general d-dimensional SDEs of the form

$$dX_t = f(X_t)\,dt + G(X_t)\,dW_t, \qquad t \in [t_0, T],$$

where $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq t_0}, \mathbb{P})$ is a filtered probability space supporting an m-dimensional standard Brownian motion $W_t = (W_t^1, \ldots, W_t^m)$. We proceed under the standing hypothesis that this SDE possesses a (pathwise) unique strong solution with almost surely positive components on $[t_0, T]$. The main contribution of this work is the development of a positivity-preserving method that achieves an almost sure pathwise convergence rate under non-standard assumptions on the coefficients of the equation, and that remarkably applies to general multidimensional SDE systems (without requiring any specific noise structure) while maintaining a computational cost essentially identical to that of the standard Euler–Maruyama scheme.

2 Proposed method and its convergence

Let $t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[t_0, T]$ with step size $h = (T - t_0)/N$, and write $\Delta W_n^j := W_{t_{n+1}}^j - W_{t_n}^j$. Given $X_n \in \mathbb{R}^d_{>0}$ approximating $X(t_n)$, define, for $t \in [t_n, t_{n+1}]$, the auxiliary process $Y(t) = (Y^1(t), \ldots, Y^d(t))^T$ solving, componentwise,

$$dY^k(t) = \frac{f^k(X_n)}{X_n^k}\,Y^k(t)\,dt + \sum_{j=1}^{m} \frac{g^{kj}(X_n)}{X_n^k}\,Y^k(t)\,dW_t^j, \qquad Y(t_n) = X_n, \tag{1}$$

$k = 1, \dots, d$, where $f^k(X_n) := f^k(X_n^1, \dots, X_n^d)$ and similarly for $g^{kj}(X_n)$.

Equation (1) admits the explicit solution, [3],

$$Y^k(t) = X_n^k \exp\left[\left(\frac{f^k(X_n)}{X_n^k} - \frac{1}{2}\sum_{j=1}^{m}\left(\frac{g^{kj}(X_n)}{X_n^k}\right)^2\right)(t - t_n) + \sum_{j=1}^{m}\frac{g^{kj}(X_n)}{X_n^k}\left(W_t^j - W_{t_n}^j\right)\right], \tag{2}$$

The numerical scheme $\{X_n\}_{n=0}^N$ approximating $\{X(t_n)\}_{n=0}^N$ is defined by $X_0 = X(t_0)$ and, for $n = 0, \dots, N-1$,

$$X_{n+1}^k = X_n^k \exp\left[\left(\frac{f^k(X_n)}{X_n^k} - \frac{1}{2}\sum_{j=1}^{m}\left(\frac{g^{kj}(X_n)}{X_n^k}\right)^2\right)h + \sum_{j=1}^{m}\frac{g^{kj}(X_n)}{X_n^k}\Delta W_n^j\right], \qquad k = 1, \dots, d, \tag{3}$$

i.e. $X_{n+1} = Y(t_{n+1})$ with Y as in (1).

The following result establishes the unconditional positivity preservation of the numerical scheme:

Proposition 2.1. *For every $h > 0$ and every realization of the driving noise, $X_0 \in \mathbb{R}^d_{>0}$ implies $X_n \in \mathbb{R}^d_{>0}$ for all $n = 0, \dots, N$.*

Proof. Immediate from (3): each X_{n+1}^k is X_n^k multiplied by the exponential of a finite real random variable, hence $X_{n+1}^k > 0$ whenever $X_n^k > 0$. The claim follows by induction. □

Regarding the rate of convergence, we establish the following result:

Proposition 2.2 (Pathwise convergence)**.** *Under suitable regularity assumptions on f, g (e.g., $f, g \in C^1$), for every $\varepsilon > 0$ there exists an a.s. finite random variable K_ε such that*

$$\max_{0 \le n \le N} \left|X(t_n) - X_n\right| \;\le\; K_\varepsilon\, h^{1/2-\varepsilon} \qquad a.s., \tag{4}$$

and, in particular, $\max_{0 \le n \le N} |X(t_n) - X_n| \xrightarrow[h \to 0]{a.s.} 0$.

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XIX ENAMA - Novembro 2026 *43–44*

NEUTRON SPECTRAL RECONSTRUCTION VIA DEEP LEARNING AND PHYSICS-INFORMED NEURAL NETWORKS A COMPARATIVE ANALYSIS

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Abstract

Neutron spectrometry using the Bonner sphere system requires solving a Fredholm integral equation of the first kind, classically defined as an ill-conditioned inverse problem. Small statistical fluctuations in the count rates induce severe instabilities in the reconstruction of the continuous energy spectrum. To overcome this limitation, this work investigates the use of deep learning as a regularization approach. We present a comparative analysis of five architectures: Multilayer Perceptron (MLP), one-dimensional Convolutional Neural Networks (1D CNN), 1D U-Net, Long Short-Term Memory (LSTM), and Physics-Informed Neural Networks (PINN). The results indicate that the 1D CNN achieved the best overall predictive performance, more effectively capturing spatial and moderation correlations between adjacent spheres.

1 Introduction

The quantification of neutron radiation is essential in metrology and reactor physics [1]. Due to the broad energy range and the electrical neutrality of neutrons, direct measurement is not feasible, requiring moderating systems such as the Bonner Sphere Spectrometer [2]. The integral response of the Bonner spheres is mathematically described by a Fredholm integral equation of the first kind:

$$A_j = \int_{E_{min}}^{E_{max}} R_j(E)\Phi(E)dE, \quad j = 1, 2, \ldots, m, \tag{1}$$

where A_j is the count rate, $R_j(E)$ is the response function, and $\Phi(E)$ is the spectral flux. Discretization yields $A_j = \sum_{k=1}^{N} \alpha_{jk}\Phi_k$. Inverting α_{jk} constitutes a Hadamard ill-posed problem [4], in which even minimal statistical fluctuations in A_j may cause unbounded nonphysical oscillations in Φ_k.

The main motivation of this study is to assess how different deep neural network architectures can act as universal approximators and regularizers [5, 6], mitigating the nonphysical oscillations generated by statistical uncertainties in the count data.

2 Main results

To map the inputs $x \in \mathbb{R}^7$ (sphere counts) to the outputs $y \in \mathbb{R}^{84}$ (spectral energy bins), the five networks were implemented using a standardized 128-64-84-neuron topology: Multilayer Perceptron (MLP) [5, 6, 7], 1D CNN, 1D U-Net [8], Long Short-Term Memory (LSTM) [9], and Physics-Informed Neural Networks (PINN) [3].

To map the inputs $x \in \mathbb{R}^7$ (sphere counts) to the output $y \in \mathbb{R}^{84}$ (spectral energy bins), five networks were implemented using the same topology (128-64-84): Multilayer Perceptron (MLP) [5, 6, 7], 1D CNN, 1D U-Net [8], Long Short-Term Memory (LSTM) [9], and Physics-Informed Neural Networks (PINN) [3]. The PINN model incorporated physical constraints directly into its loss function, defined as $\mathcal{L}_{total} = \mathcal{L}_{data} + \lambda\mathcal{L}_{physics}$. In this context, the term $\mathcal{L}_{physics}$ acts as a regularizer against abrupt derivatives and nonphysical variations in the continuity of the

neutron flux, ensuring a mathematically smooth solution. The hyperparameter λ controls the relative importance of this physical constraint with respect to the purely statistical error ($\mathcal{L}_{data}$).

The dataset, consisting of 286 simulations for each of the 84 spectra, was divided into 70% for training and 30% for testing. Mean Squared Error (MSE) was adopted as the main regression metric [10]. Table 1 presents the consolidated metrics.

Model	MLP	1D CNN	U-Net	LSTM	PINN
MSE (Loss)	0.022274	0.012879	0.026780	0.029058	0.032848
MAE	0.065452 %	0.050732 %	0.070516 %	0.086083 %	0.081805 %

Table 1: Model performance on the test set.

The regression metrics indicate that the 1D CNN achieves the best overall performance (MSE = 0.012879, MAE = 0.050732), reflecting the advantage of convolutional operations in extracting adjacent topological dependencies. Although the PINN yielded a slightly higher MSE (0.032848), this reflects a beneficial physics-based trade-off: imposing mathematical regularity limits statistical overfitting in favor of a physically smooth and consistent solution. The U-Net also demonstrated good ability to preserve fine spectral boundaries. Figure 1 illustrates the predictive capabilities.

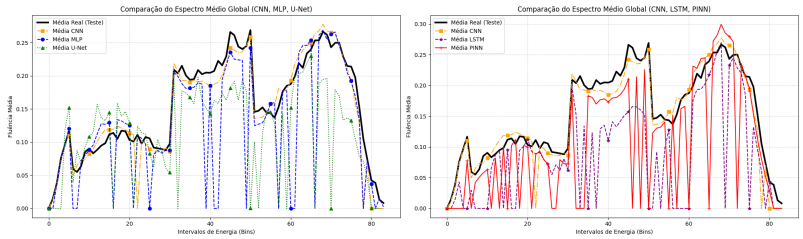


Figure 1: Comparison of the global mean spectrum for the test set.

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XIX ENAMA - Novembro 2026 *45–46*

EXTREME POINT MAXIMIZATION OF CONVEX QUADRATIC RESPONSE SURFACES AND EXACT LEAVE-ONE-OUT VALIDATION

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Abstract

Response Surface Methodology maximizes a fitted quadratic $\widehat{Y}$ over a coded box $Q = [-1, 1]^n$, a problem we treat by convex analysis rather than by numerical search. Since the task is NP-hard in general, no certificate of global optimality follows from local information alone. We show that a positive definite least squares Hessian confines the maximizer to $\mathrm{ext}(Q)$, turning the search into a finite comparison with zero gap, and that a rank-one update gives the leave-one-out residuals in closed form, making surrogate validation exact rather than resampled. The design of $NiCo_2S_4$–graphene supercapatteries illustrates both.

1 Introduction

Response Surface Methodology fits a second-order polynomial to a small factorial design and maximizes it over the coded region [1], which raises two questions of analysis. The first is certification, since a stationary point or a metaheuristic run returns a candidate with no bound separating it from the unknown global value, and the generic remedy, convex relaxations tightened until their bounds meet [2], costs what the NP-hardness of the problem requires [3]. The second is model choice, since several surrogates reproduce the same scarce data while the in-sample residual stops measuring prediction. Structure replaces search by closed form in both. With $Q = [-1, 1]^n$ and $\widehat{Y}(x) = x^\top Ax + b^\top x + c$, A symmetric, we consider

$$\max_{x \in Q} \widehat{Y}(x), \tag{1}$$

where A, b and c are least squares estimates from the 20 runs of the central composite design of [4] for $NiCo_2S_4$–graphene hybrids, whose coded factors are the graphene/NCS mass ratio, the hydrothermal time and the S/Ni molar ratio, with capacitance (F/g) as response. The fit has $R^2 = 0.9716$ and reads

$$\begin{aligned} \widehat{Y}(x) &= 908.75 + 426.50\,x_1 + 1.83\,x_2 + 152.78\,x_3 + 37.75\,x_1x_2 + 133.00\,x_1x_3 \\ &+ 137.00\,x_2x_3 + 65.65\,x_1^2 + 228.24\,x_2^2 + 171.69\,x_3^2. \end{aligned} \tag{2}$$

2 Main Results

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be nonempty, compact and convex, and let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and continuous. Then f attains its maximum over K at an extreme point of K [5].*

Proof Compactness and continuity give a maximizer $\bar{x} \in K$, and the Minkowski theorem writes $\bar{x} = \sum_{i=1}^m \lambda_i v_i$ with $v_i \in \mathrm{ext}(K)$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, so convexity gives $f(\bar{x}) \leq \sum_i \lambda_i f(v_i) \leq \max_i f(v_i)$. Since $\mathrm{ext}(K) \subset K$, some v_i maximizes f. ∎

Corollary 2.1. *Let A be positive definite and let $g(s) = \widehat{Y}(-\mathbf{1} + 2s)$. Then*

$$\max_{x \in Q} \widehat{Y}(x) = \max_{v \in \{-1,1\}^n} \widehat{Y}(v) = \max_{s \in \{0,1\}^n} g(s) = \max_{s \in [0,1]^n} g(s). \tag{1}$$

Proof The Hessian of $\widehat{Y}$ is $2A$, so $\widehat{Y}$ is convex on the compact polytope Q, whose extreme points are the sign vectors, and Theorem 2.1 gives the first equality. The affine bijection $s \mapsto -\mathbf{1} + 2s$ maps $[0,1]^n$ onto Q and $\{0,1\}^n$ onto $\mathrm{ext}(Q)$, so g is convex and the remaining equalities follow from Theorem 2.1 applied to g. ∎

Problem (1) is thus settled by 2^n comparisons, with an exact binary relaxation. For (2) the eigenvalues of $2A$ are 65.3, 291.2 and 574.7, so eight comparisons return 2263.19 F/g at $x_1 = x_2 = x_3 = 1$, the corner being preferred to the unique critical point $-\frac{1}{2}A^{-1}b$, a minimizer. The last equality in (1) is what a mixed-integer quadratic solver exploits, certifying the same value at relative gap 4×10^{-16}, whereas local solvers and metaheuristics return the point with no dual bound. The second question is equally explicit, with X the design matrix, of full column rank p, and $H = X(X^\top X)^{-1}X^\top$ its projector.

Proposition 2.1. *Let $e_i = y_i - \hat{y}_i$ be the residuals of the fit on all N observations and $\hat{y}_i^{(-i)}$ the prediction at x_i of the model fitted without the i-th one. If $h_{ii} < 1$, then $y_i - \hat{y}_i^{(-i)} = e_i/(1 - h_{ii})$.*

Proof Deleting the i-th row gives the rank-one update $X_{(-i)}^\top X_{(-i)} = X^\top X - x_i x_i^\top$, whose inverse follows from the Sherman–Morrison formula since $1 - h_{ii} \neq 0$. Substituting it into the normal equations of the deleted fit and evaluating at x_i gives $\hat{y}_i^{(-i)} = \hat{y}_i - h_{ii}e_i/(1 - h_{ii})$. ∎

Leave-one-out validation is therefore a deterministic functional of one fit, and the leverages quantify the scarce data regime, since $\sum_i h_{ii} = p$ forces an average leverage p/N, here 10/20. For (2) it gives $R^2_{\mathrm{LOO}} = 0.843$, against 0.645 for support vector regression and less than 0.42 for tree ensembles, all of which exceed the quadratic in sample by interpolating the design points. Corollary 2.1 thus certifies the optimizer and Proposition 2.1 selects the surrogate, both in closed form. Code is at `github.com/thiagoferrobr/opt`.

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XIX ENAMA - Novembro 2026 *47–48*

CONTROLLING THE CONDITION NUMBER OF MULTIQUADRIC RBF MATRICES VIA POISSON DISK SAMPLING

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Abstract

Multiquadric (MQ) radial basis function (RBF) interpolation generates ill-conditioned matrices. Using Poisson Disk Sampling (with minimum distance h) and spectral analysis, we derive $\kappa(A)$ and show that minimization yields $r_0^* = c\sqrt{2}$, with $\kappa_{\min} = 3N - 1$, for quasi-uniform points. For unstructured points, the optimality condition yields a constant $\kappa = 1 + 2K$, independent of N. Numerical experiments confirm that the adaptive choice of $r_{\min}(N)$ keeps $\kappa(A)$ bounded and growing much more slowly than the fixed and quasi-uniform alternatives.

1 Introduction

For the MQ $\phi(r) = \sqrt{r^2 + c^2}$, the matrix $A_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ is ill-conditioned when the points are close to one another. To control this problem, we impose a distribution of the centers using *Poisson Disk Sampling* [2], which guarantees $\|\mathbf{x}_i - \mathbf{x}_j\| \geq h$ for $i \neq j$, thereby preventing clustering. We minimize $\kappa(A)$ subject to this constraint.

2 Spectral Analysis

2.1 Quasi-uniform points

Expanding $\phi(r) = c + \frac{r^2}{2c} - \frac{r^4}{8c^3} + \cdots$ and approximating $R^{(2)} \approx r_0^2(J - I)$, $R^{(4)} \approx r_0^4(J - I)$, we obtain $A \approx \alpha I + \beta J$, where $\alpha = -\frac{r_0^2}{2c} + \frac{r_0^4}{8c^3}$ and $\beta = c + \frac{r_0^2}{2c} - \frac{r_0^4}{8c^3}$.

Lemma 2.1. *For $A = \alpha I + \beta J$, the eigenvalues are $\lambda_1 = \alpha + \beta N$ and $\lambda_i = \alpha$ for $i \geq 2$.*

Thus,

$$\kappa(A) \approx \frac{\left|cN + \frac{N-1}{2c} r_0^2 - \frac{N-1}{8c^3} r_0^4\right|}{\left|-\frac{r_0^2}{2c} + \frac{r_0^4}{8c^3}\right|}. \tag{1}$$

Under $0 < r_0 \ll c \ll N$, the denominator is negative and the numerator is positive, and

$$\kappa(A) \approx \frac{cN + \frac{N-1}{2c} r_0^2 - \frac{N-1}{8c^3} r_0^4}{\frac{r_0^2}{2c}\left(1 - \frac{r_0^2}{4c^2}\right)}. \tag{2}$$

Proposition 2.1. *Minimizing (2) yields $r_0^* = c\sqrt{2}$ and, by substitution into (1), $\kappa_{\min} = 3N - 1$.*

2.2 Unstructured points

For unstructured points, perturbation theory yields,

$$\kappa(N) = 1 + \frac{2Nc^2}{(N-1)\sigma_{\text{eff}}^2}, \quad \text{where} \quad \sigma_{\text{eff}}^2 = \mu_{\text{r}}^2 + \sigma_{\text{r}}^2.$$

Proposition 2.2. *If $\frac{d\kappa}{dN} = 0$, then $\frac{c^2}{\sigma_{eff}^2} = K\frac{N-1}{N}$ and $\kappa = 1 + 2K$, a constant independent of N. This limit is asymptotic, valid when $N \to \infty$ and $\frac{r_{\min}}{c} \to 0$.*

3 Experiments

Simulations on the fixed domain $[0, 30]^2$, with $c = 0.15$, compare three strategies: (i) fixed $r_{\min} = 0.005$; (ii) $r_{\min} = c\sqrt{2}$; and (iii) $r_{\min}(N) = \frac{c}{\sqrt{K}}\sqrt{\frac{N}{N-1}}$, with $K = e^{-2.9}$. For each N, it is verified that all N points are effectively generated. The condition number was computed by SVD in double precision.

Table 1: $\kappa(A)$ for selected values of N and theoretical values.

N	$r_{\min} = 0.005$	$r_{\min} = c\sqrt{2}$	Theor. $3N-1$	$r_{\min}(N)$	Theor. $1+2K$
5	1.41×10^1	3.36×10^1	14	1.69×10^1	1.11
505	3.49×10^5	1.49×10^5	1514	3.61×10^4	1.11
1005	1.13×10^7	2.96×10^5	3014	8.66×10^4	1.11
1505	7.95×10^7	8.39×10^5	4514	1.46×10^5	1.11

Table 1 shows that fixed $r_{\min}$ produces severe ill-conditioning (κ ranging from 10^1 to 10^7), the strategy $r_{\min} = c\sqrt{2}$ grows linearly (Proposition 1), and the adaptive strategy $r_{\min}(N)$ keeps κ bounded (Proposition 2). The observed values lie above $1+2K$ because the Taylor expansion $\sqrt{r^2+c^2} \approx c + \frac{r^2}{2c}$ is valid only for $r_0 \ll c$ (here $\frac{r_{\min}}{c} \approx \frac{1}{\sqrt{K}} \approx 4.26$), and because the optimality condition is derived for $\sigma^2_{\text{eff}} = \mu_r^2 + \sigma_r^2$, whereas the experiments constrain only $r_{\min}$; since $\mu_r > r_{\min}$ and $\sigma_r^2 > 0$, the effective distance exceeds $r_{\min}$.

4 Conclusion

The adaptive choice $r_{\min}(N) = \frac{c}{\sqrt{K}}\sqrt{\frac{N}{N-1}}$ keeps $\kappa(A)$ bounded and growing much more slowly than the fixed and quasi-uniform alternatives, although it does not attain the theoretical value $\kappa = 1 + 2K$. This limit is asymptotic, valid when $N \to \infty$ and $\frac{r_{\min}}{c} \to 0$, but here $\frac{r_{\min}}{c} \approx \frac{1}{\sqrt{K}} \approx 4.26$, so the Taylor expansion truncated after the quadratic term loses accuracy and higher-order terms increase κ. In addition, the optimality condition is derived for $\sigma^2_{\text{eff}} = \mu_r^2 + \sigma_r^2$, whereas the experiments constrain only $r_{\min}$; since $\mu_r > r_{\min}$ and $\sigma_r^2 > 0$, the effective distance exceeds $r_{\min}$. Even so, an appropriate choice of $r_{\min}$ is essential for the stable use of multiquadric interpolation.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *49–50*

OPTIMIZATION OF THE POLYMER INJECTION PROCESS FOR IMPROVED OIL RECOVERY

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Abstract

Fossil fuels, particularly oil, remain a primary component of the global energy matrix, accounting for over 30%. For this reason, oil recovery processes are of both research and economic interest, especially because roughly 70% of the total oil volume remains in the reservoir after the initial recovery stages. To mobilize this bypassed oil, Improved Oil Recovery (IOR) methods are employed. Among them, polymer injection stands out as one of the main techniques. The objective of this study is to present an optimization problem for the injection of polymer during improved oil recovery process using automatic differentiation.

1 Introduction

During a waterflood, an unfavorable mobility ratio leads to viscous fingering, a phenomenon in which water bypasses the oil and produces an inefficient sweep. To mitigate this, polymer is added to the injected water to raise its viscosity and improve the mobility ratio. The resulting displacement can be modeled as a one-dimensional, two-phase immiscible flow in a porous medium, derived from mass conservation under the following assumptions: the flow is linear and horizontal, both phases are incompressible and immiscible, and capillary and gravitational forces are negligible. In this context, the mass conservation equations for the wetting and non-wetting phases, together with the pressure equation obtained by adding them, are given by:

$$\frac{\partial}{\partial x}\left(\lambda_w \frac{\partial p}{\partial x}\right) = \phi \frac{\partial S_w}{\partial t} + q_w, \quad \frac{\partial}{\partial x}\left(\lambda_o \frac{\partial p}{\partial x}\right) = \phi \frac{\partial S_o}{\partial t} + q_o, \quad \frac{\partial}{\partial x}\left(\lambda_T \frac{\partial p}{\partial x}\right) = q_w + q_o \tag{1}$$

where $\lambda_T = \lambda_w + \lambda_o$ is the total mobility, λ_w is the mobility of the aqueous phase, λ_o is the mobility of the oil, λ_T is the total mobility given by $(\lambda_T = \lambda_w + \lambda_o)$, q_w and q_o are the source terms of each phase. However, a simplified one-dimensional formulation will be used for the polymer transport in two-phase flow, assuming that the polymer acts only in the wetting phase, disregarding adsorption, the chemical degradation reaction, and dispersion.

Furthermore, it is important to describe how the polymer alters the viscosity of the aqueous phase. Because water consists of small molecules while the polymer is a long-chain macromolecule, dissolving polymer in the injected water substantially increases its viscosity. A higher aqueous viscosity μ_w lowers the water mobility $\lambda_w = k_{rw}/\mu_w$, which mitigates viscous fingering and improves sweep efficiency. Following [1], the polymer modifies the aqueous-phase mobility by replacing the water viscosity μ_w with the polymer-solution viscosity $\eta(c)$, which depends on the polymer concentration c:

$$\bar{\lambda}_w = \frac{k\, k_{rw}}{\eta(c)}. \tag{2}$$

Further details on the numerical discretization, physical parameters and modelling assumptions can be found in [1].

Discretizing the model (1) under the IMPES scheme and specifying the reservoir parameters yields the pressure, saturation, and polymer concentration fields over time. From these, the cumulative oil production can be computed as a function of the polymer injection time. This motivates the definition of a profit function:

$$F = \left(\frac{P}{V}\right)\alpha - \tau\, t_i \tag{3}$$

where P is the cumulative oil production, V the volume of a single barrel, α the price per barrel, τ the polymer cost per unit injection time, and t_i the polymer injection time. The term $(P/V)\alpha$ represents the revenue from the recovered oil, while $\tau\, t_i$ is the cost of the injected polymer, so F expresses the net profit as a function of the injection time. The goal is to determine the injection time t_i that maximizes (3). To this end, the numerical model is implemented in JAX [2], so that the gradient of F with respect to t_i is obtained through automatic differentiation rather than finite-difference approximations. The `value_and_grad` function from JAX returns both F and its gradient, which are supplied to the *minimize* routine of *SciPy* applied to $-F$. This yields the optimal polymer injection time at which profit is maximized.

2 Main Results

Figure 1 presents the main results in terms of saturation and the optimal polymer injection time that yields maximum profit. For an 800-day simulation, the optimization identified a maximum profit at an optimal polymer injection time of 122 days (at 500 ppm concentration)*, as illustrated in Figure 1.

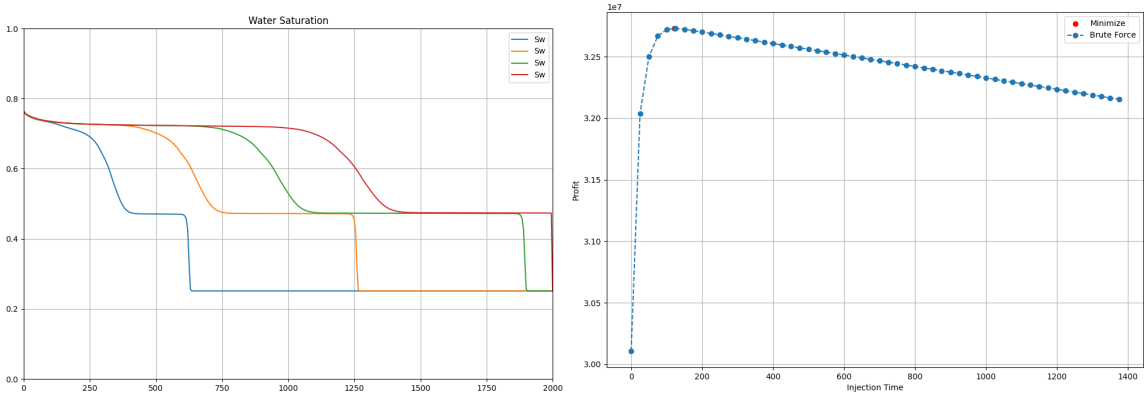


Figure 1: Left: saturation profiles. Right: optimal polymer injection time.

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XIX ENAMA - Novembro 2026 *51–52*

FRACTIONAL SIR MODEL WITH CROSS DIFFUSION

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Abstract

In this work, we present the results of a study on the application of the Caputo derivative to an SIR model with cross-diffusion. The primary focus is on the numerical aspect of the problem, for which we propose methods to compute the solution involving the Caputo derivative. Based on this, we conduct a qualitative analysis of the solution, interpreting the changes introduced by the Caputo derivative into the model.

1 Introduction

The objective of this work is to apply the tools of fractional calculus to the SIR model with cross-diffusion proposed in [5]. For this purpose, we consider the time derivative to be of an arbitrary order $\alpha \in (0, 1]$ in the sense of the Caputo derivative, which is denoted by ${}^{C}D^{\alpha}$. The Caputo fractional derivative is defined as follows:

$$\left({}^{C}D^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad t > 0.$$

Further information on fractional calculus can be found in [1]. We then consider this derivative in the SIR model, which represents the spread of an infectious disease through the interaction between susceptible (S) and infected (I) individuals. When cross-diffusion is added, the spatial propagation of the disease is also accounted for. The fractional version of the model is given by:

$${}^{C}D^{\alpha} S = D_{11}\Delta S + D_{12}\Delta I - \beta SI, {}^{C}D^{\alpha} I = D_{21}\Delta S + D_{22}\Delta I + \beta SI - \gamma I,$$

where D_{11}, D_{12}, D_{21}, and D_{22} are diffusion coefficients. The reaction terms are $-\beta SI$ and $\beta SI + \gamma I$, where β is the infection rate and γ is the removal rate from the infected compartment.

The Caputo derivative is an operator that considers a function's entire history, making it widely used in models to incorporate memory effects into equations. However, due to this characteristic, the numerical cost of discretizing the operator is higher than that of the classical derivative. Consequently, numerical methods that can provide a good approximation of the solution, even with coarser time-stepping, are particularly useful in this context.

We then adopted a numerical approach to the problem based on the schemes presented in [2], with the aim of obtaining methods that would provide good approximations of the solution at reasonable computational cost. Subsequently, we conducted a qualitative analysis of the numerical solutions, examining the main changes introduced to the model by using the fractional derivative.

2 Main Results

The numerical methods employed produced consistent results when compared with each other and with the classical case when $\alpha = 1$. For example, we consider the values $D_{11} = 0.8$, $D_{12} = 0.7$, $D_{21} = 0.3$, $D_{22} = 0.4$, $\beta = 0.2$, and $\gamma = 0.08$. Figure (1) illustrates the spatial distribution of infected individuals at various time points and for different values of α.

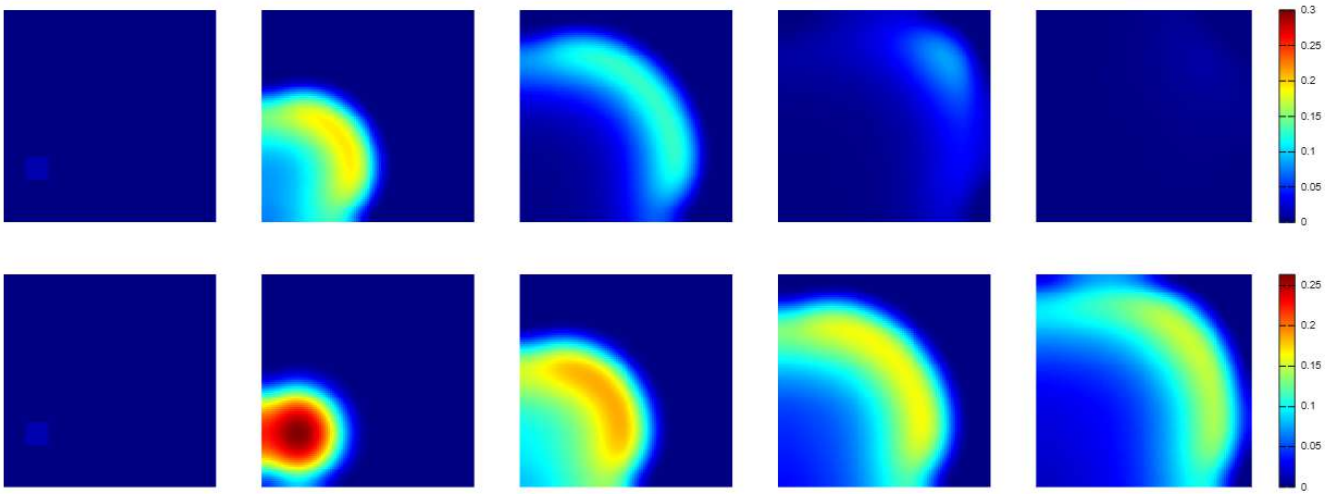


Figure 1: The graphs show the concentration of infected individuals at times $t \in \{0, 100, 200, 300, 400\}$ for $\alpha = 1$ (top) and $\alpha = 0.8$ (bottom).

From the solution, we can see that reducing the value of α results in delayed disease transmission and an increased number of infected individuals. Consequently, the model is expected to more accurately represent real-world infectious disease scenarios, in line with the discussions presented in references [3, 4], while also contributing to the study of fractional modeling.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *53–54*

ON GROWTH RATE COMPARISONS: μ-DICHOTOMY, BOUNDED GROWTH, AND SPECTRAL PROPERTIES

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Abstract

We analyze the dichotomy spectrum of nonautonomous linear systems for general growth regimes. Through newly introduced comparison criteria, we unravel the interaction between bounded growth and generalized dichotomies. In particular, we highlight the spectral evolution under these benchmarks, showing that faster growth rates contract the spectrum, while slower rates foster its expansion.

1 Introduction

The classical exponential dichotomy spectrum—also known as the Sacker & Sell spectrum—often fails to provide meaningful information when nonautonomous linear systems exhibit non-standard growth. For example, the continuous system

$$\dot{x}(t) = 2|t|x(t), \qquad t \in \mathbb{R}, \tag{1}$$

as well as its discrete analogue $x(k+1) = A(k)x(k)$ with $A(k) = e^{2k+1}$ for $k \geq 0$ and $A(k) = e^{-2k-1}$ for $k < 0$, yield an exponential dichotomy spectrum equal to $\{+\infty\}$. This uninformative spectrum places such systems outside the scope of many results that rely on conditions involving the dichotomy spectrum.

By replacing the classical exponential rate with a quadratic exponential rate $t \mapsto e^{\operatorname{sgn}(t)t^2}$, system (1) exhibits quadratic exponential bounded growth and its dichotomy spectrum compresses to the single point $\{1\}$, illustrating how a suitable growth rate yields a far more refined description of the dynamics.

Motivated by this behavior, we establish a rigorous comparison framework for general growth rates (μ and ω) to investigate the interplay between generalized μ-dichotomies, bounded growth, and spectral evolution:

- **Interplay of Dichotomy and Bounded Growth:** Introducing a strong comparison notion ($\mu \gg \omega$, where μ is faster than ω), we establish that the existence of a μ-dichotomy prevents the system from exhibiting ω-bounded growth for any slower rate ω. Conversely, ω-bounded growth precludes the existence of a μ-dichotomy for any faster rate μ.

- **Spectral Evolution under Weak Comparisons:** Under a weaker comparison relation ($\mu \succ \omega$), we analyze how the dichotomy spectrum transforms across different growth regimes. Specifically, we prove that faster growth rates compress the spectrum toward zero, whereas slower growth rates force the spectrum to expand toward $\pm\infty$.

2 Main Results

Consider the nonautonomous linear system

$$x'(t) = A(t)x(t), \qquad t \in \mathbb{R}, \tag{1}$$

where $A\colon \mathbb{R} \to \mathcal{M}_d(\mathbb{R})$ is locally integrable. The **μ-dichotomy spectrum** of (1) is the set

$$\Sigma_\mu D(A) := \{\gamma \in \mathbb{R} \cup \{\pm\infty\} : \text{ the } (\mu,\gamma)\textbf{-shifted} \text{ of (1) does not admit } \mu D\}.$$

The main results of our work [1] are the following:

Theorem 2.1. *Consider two growth rates μ, ω such that μ is faster than ω. Suppose that system (1) has μD. Then, its ωD spectrum is either $\{+\infty\}$, $\{-\infty\}$ or $\{\pm\infty\}$.*

Theorem 2.2. *If the system (1) has ω-bounded growth and μ is faster than ω, then $\Sigma_\mu D(A) = \{0\}$.*

Theorem 2.3. *Consider two growth rates $\mu, \omega : \mathbb{Z} \to \mathbb{R}^+$ such that μ is weakly faster than ω. Fix $a > 0$. Then, for system (1):*

i) If $\Sigma_\mu D(A) \subset [-\infty, -a]$, then $\Sigma_\omega D(A) \subset [-\infty, -a]$,

ii) If $\Sigma_\mu D(A) \subset [a, +\infty]$, then $\Sigma_\omega D(A) \subset [a, +\infty]$.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *55–56*

ON STABILITY FOR LINEAR MEASURE FUNCTIONAL DIFFERENTIAL EQUATIONS THROUGH GENERALIZED LINEAR EQUATIONS

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Abstract

We investigate the stability of linear measure functional differential equations (MFDEs) through their correspondence with generalized linear differential equations (GLDEs). A suitable phase space is introduced to establish a one-to-one correspondence between global forward solutions of both equations for arbitrary initial times. This framework provides a direct relation between the evolution family of the MFDE and the fundamental operator of the associated GLDE. As a consequence, recent stability results for GLDEs yield characterizations of uniform, asymptotic and exponential stability for linear MFDEs. We also present examples illustrating that some stability properties are not preserved under the correspondence, thereby clarifying its scope and limitations.

1 Introduction

Linear MFDEs can be analyzed through generalized linear differential equations (GLDEs), allowing the use of tools from the theory of generalized ODEs. A major difficulty in extending this correspondence to global forward solutions is the compatibility between the initial conditions of both problems. This requires the construction of an appropriate phase space in which the associated GLDE is well posed.

In this work, we introduce such a phase space and establish a one-to-one correspondence between global forward solutions of linear MFDEs and GLDEs. We then relate the evolution family of the MFDE to the fundamental operator of the GLDE and apply recent stability results for generalized linear differential equations to characterize several stability properties of linear MFDEs.

2 Main Results

The main contribution of this work is the construction of a suitable phase space for generalized linear differential equations (GLDEs), allowing the study of linear measure functional differential equations (MFDEs) through a precise correspondence between their global forward solutions.

For each $s_0 \geq 0$, define the extension operator

$$[\widetilde{y}_{s_0}(t)](\theta) = \begin{cases} y(s_0 - r), & -r \leq \theta \leq s_0 - r, \\ y(\theta), & s_0 - r \leq \theta \leq t, \\ y(t), & t \leq \theta < \infty, \end{cases}$$

and introduce the phase space

$$\mathcal{O} = \bigcup_{s_0 \geq 0} \mathcal{O}_{s_0}, \qquad \mathcal{O}_{s_0} = \bigcup_{t \geq s_0} \mathcal{O}_{t,s_0},$$

where

$$\mathcal{O}_{t,s_0} = \{\widetilde{y}_{s_0}(t) : y \in BG([-r, \infty), \mathbb{R}^d)\}.$$

The space $\mathcal{O}$ encodes the compatibility conditions between the initial data of the MFDE and the associated GLDE and provides the appropriate functional framework for establishing the correspondence between both equations.

Our first result establishes the correspondence theorem. This result allows the dynamics of the MFDE to be represented through the generalized linear equation and serves as the basis for the subsequent stability analysis. Next, we relate the evolution family naturally associated with the MFDE to the fundamental operator of the GLDE.

Theorem 2.1 (Operator Correspondence)**.** *Let $T(t,s)$ denote the evolution family generated by the linear MFDE and let $U(t,s)$ be the fundamental operator of the corresponding GLDE. Then, for every admissible initial condition,*

$$T(t,s)\varphi = \big(U(t,s)\widetilde{\varphi}\big)_t,$$

where $\widetilde{\varphi}$ denotes the compatible extension of the initial history.

This identity establishes a direct connection between the dynamics of both systems and enables the transfer of stability results from generalized differential equations to measure functional differential equations. As an application of this framework, sufficient conditions for stability are obtained.

Theorem 2.2. *Assume that the coefficient function satisfies an appropriate boundedness condition with respect to the Kurzweil–Stieltjes integral. Then the trivial solution of the associated GLDE is stable (respectively, uniformly stable).*

Finally, by combining the operator correspondence with recent stability results for generalized linear differential equations, we derive several characterizations of stability for linear MFDEs.

By combining the operator correspondence with recent stability results for generalized linear differential equations, we obtain characterizations of uniform stability, uniform asymptotic stability, and global uniform exponential stability in terms of the associated evolution operators.

An explicit example is also presented showing that, although the correspondence preserves several stability properties, global asymptotic stability and global uniform exponential stability are not preserved in general. This example highlights fundamental differences between the phase spaces of MFDEs and GLDEs and clarifies the limitations of the correspondence.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *57–58*

TOPOLOGICAL METHODS IN THE STUDY OF THE EXISTENCE OF AT LEAST ONE PAIR OF NONNEGATIVE PERIODIC SOLUTIONS FOR A PARTICULAR CLASS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The aim of this work is to investigate the existence of at least two nonnegative periodic solutions for a class of nonlinear second-order differential equations of the form

$$u'' + c(t)u' + \lambda a(t)g(u) = 0, \tag{1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a T-periodic integrable function, c is a T-periodic continuous function and $g : [0, \infty) \to [0, \infty)$ is a given function. The approach was primarily topological, based on degree theory and the method of upper and lower solutions.

1 Introduction

In [1], the authors study the nonlinear second-order differential equation

$$u'' + cu' + \lambda a(t)g(u) = 0, \tag{2}$$

subject to either periodic or Neumann boundary conditions, where $c \in \mathbb{R}$. Their work is concerned with the existence and multiplicity of positive periodic solutions.

The authors assume the following hypotheses: $\lambda > 0$; (a) is a real-valued, locally integrable, (T)-periodic function satisfying

$$\int_0^T a(t), dt < 0,$$

and there exists an interval $I \subseteq [0, T]$ such that $a(t) \geq 0$ for almost every $t \in I$ and

$$\int_I a(t), dt > 0;$$

moreover, g is a positive continuous function defined on $[0, +\infty)$, which is superlinear at zero, sublinear at infinity, and regularly oscillating at zero.

The authors prove the existence of at least two positive T-periodic solutions of equation (2) for sufficiently large values of λ. Their approach relies mainly on Mawhin's coincidence degree theory together with fixed point index results for certain Poincaré translation operators (also known as first return operators).

2 Main Results

In our setting, we assume the following hypotheses:

(1) c is a continuous T-periodic function defined on $\mathbb{R}$;

(2) a is a real-valued, T-periodic, integrable function satisfying:

(a^*) $\int_0^T e^{\int_0^t c(s)\,ds} a(t)\,dt < 0$;

(a^{**}) There exists an interval $I \subseteq [0,T]$ such that $a(t) \geq 0$ for almost every $t \in I$ and

$$\int_I e^{\int_0^t c(s)\,ds} a(t)\,dt > 0.$$

(3) g is a continuous function defined on $[0,+\infty)$ satisfying:

(g_{**}) $g(0) = 0$ and $g(s) > 0$ for every $s > 0$;

(gs_0) $\lim_{s\to 0^+} \frac{g(s)}{s} = 0$;

(gs_∞) $\lim_{s\to +\infty} \frac{g(s)}{s} = 0$;

(O_0) $\lim_{\substack{s\to 0^+ \\ \omega\to 1}} \frac{g(\omega s)}{g(s)} = 1$;

(O_∞) $\lim_{\substack{s\to +\infty \\ \omega\to 1}} \frac{g(\omega s)}{g(s)} = 1$.

(4) $\lambda > 0$.

Our main result can be stated as follows.

Theorem 2.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$, $a : \mathbb{R} \to \mathbb{R}$, and $c : \mathbb{R} \to \mathbb{R}$ satisfy the assumptions stated above. Then there exists $\lambda^* > 0$ such that, for every $\lambda > \lambda^*$, equation (1) admits at least two positive (T)-periodic solutions.*

A key step toward achieving our goal was to establish weak and strong maximum principles for equation (1). This was accomplished by proving the following results, which, to the best of our knowledge, are not available in the existing literature.

Theorem 2.2 (Weak Maximum Principle)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function that is T-periodic with respect to the first variable, for some positive constant T. Assume that $f(t,s) > 0$ whenever $s < 0$, for almost every $t \in \mathbb{R}$. Then every T-periodic solution of the equation*

$$u'' + c(t)u' + f(t,u(t)) = 0, \tag{3}$$

where $c : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function, is nonnegative.

Theorem 2.3 (Strong Maximum Principle)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function that is T-periodic with respect to the first variable, for some positive constant T. Assume that there exists a function $q \in L^1([0,T])$ such that $|f(t,s)| \leq q(t)s$, for almost every $t \in [0,T]$ and every $s \in \mathbb{R}^+$. Then every nontrivial, nonnegative T-periodic solution of the equation*

$$u'' + c(t)u' + f(t,u(t)) = 0, \tag{4}$$

where $c : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function, satisfies $u(t) > 0$, for all $t \in \mathbb{R}$.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *59–60*

ON (θ, T)-PERIODIC SOLUTIONS AND FLOQUET THEORY FOR ABSTRACT GENERALIZED ODES

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Abstract

It is known that generalized ordinary differential equations (generalized ODEs for short) encompass other types of equations such as impulsive differential equations as well as dynamic equations on time scales. The present work concerns the theory of (θ, T)-periodic solutions in the framework of generalized ODEs in Banach spaces. We exhibit necessary and sufficient conditions for a solution of a generalized ODE to be (θ, T)-periodic. Moreover, we develop the Floquet theory of homogeneous linear generalized ODEs and, as a consequence, we present a characterization of fundamental matrices for the finite dimensional case.

1 Introduction

The periodicity theory is an important well-established branch of the modern theory of differential equations. Besides, it is a theory that arises in many applied problems of physics, biology, economy, medicine, technology and others. A function f defined on $\mathbb{R}$ is called to be (θ, T)-periodic if $f(t+T) = \theta f(t)$ for every $t \in \mathbb{R}$. In the particular case when $\theta = 1$, a $(1, T)$-periodic function is a classical T-periodic function, and when $\theta = -1$, a $(-1, T)$-periodic function becomes a T-antiperiodic function. The aim of this work is to study (θ, T)-periodicity of solutions of generalized ODEs of type

$$\frac{dx}{d\tau} = DF(x, t) \tag{1}$$

in a Banach space X, where $F : X \times J \to X$ is a function and $J \subset \mathbb{R}$ is an interval such that $\sup J = \infty$. Furthermore, we present a version of the Floquet Theorem for the linear case.

2 Main Results

First, we introduce the concept of (θ, T)-periodicity in the framework of generalized ODEs.

Definition 2.1. *A function $f : J \to X$ is said to be (θ, T)-periodic if there exist constants $T > 0$ and $\theta \in \mathbb{C} \setminus \{0\}$ such that*

$$f(t + T) = \theta f(t) \quad \text{for all} \quad t \in J.$$

Next, we establish a necessary and sufficient condition for a solution of the generalized ODE (1) to be (θ, T)-periodic.

Proposition 2.1. *Let $s_0 \in J$ and $x : [s_0, \infty) \to X$ be a solution of the generalized ODE (1). Then x is (θ, T)-periodic if and only if $(1 - \theta)x(t) + \int_t^{t+T} DF(x(\tau), s) = 0$, for all $t \in [s_0, \infty)$.*

Under the assumption of existence and uniqueness of solutions (condition (EU)), the following theorem provides a sufficient condition for (θ, T)-periodicity.

Theorem 2.1. *Let $s_0 \in J$, $T > 0$, $\theta \in \mathbb{C} \setminus \{0\}$ and $x : [s_0, \infty) \to X$ be a solution of the equation (1). Assume that there exists an operator $C : X \to X$ such that*

$$\theta^{-1} F(\theta x, t + T) - F(x, t) = C(x),$$

for all $x \in X$ and $t \in J$. If F satisfies condition (EU) and $x(s_0 + T) = \theta x(s_0)$, then x is (θ, T)-periodic.

We consider the homogeneous linear generalized ODE

$$\frac{dx}{d\tau} = D[A(t)x], \tag{1}$$

where $A \in BV_{loc}(J, L(X))$ satisfies $A(t+T) - A(t) = C_0$ for some $C_0 \in L(X)$, alongside regular backward difference conditions. Let $U(t, s)$ be the associated fundamental operator.

Theorem 2.2 (Periodicity of the Fundamental Operator)**.** *Under the previous assumptions, the fundamental operator $U : \mathcal{P} \to L(X)$ satisfies*

$$U(t + T, s + T) = U(t, s)$$

for all $(t, s) \in \mathcal{P}$.

Theorem 2.3 (Floquet Theorem in $\mathbb{R}^n$)**.** *Let $U : \mathcal{P} \to L(\mathbb{R}^n)$ be the fundamental operator associated with (1). Then, there exist a T-periodic matrix-valued function $P : J \to L(\mathbb{C}^n)$ and a matrix $B \in L(\mathbb{C}^n)$ such that*

$$U(t, s) = P(t) e^{B(t-s)} P^{-1}(s)$$

for all $(t, s) \in \mathcal{P}$. Equivalently, there exist a real matrix $D \in L(\mathbb{R}^n)$ and a $2T$-periodic real matrix-valued function $Q : J \to L(\mathbb{R}^n)$ such that $U(t, s) = Q(t) e^{D(t-s)} Q^{-1}(s)$.

Proposition 2.2. *A number $\theta \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of the monodromy operator $U(s_0 + T, s_0)$ if and only if the homogeneous system (1) has a non-trivial (θ, T)-periodic solution.*

Theorem 2.4. *The nonhomogeneous linear equation*

$$\frac{dx}{d\tau} = D[A(t)x + g(t)],$$

where $g(t + T) - \theta g(t) = C$, has a unique (θ, T)-periodic solution if and only if the operator $U(s_0 + T, s_0) - \theta I$ admits a left inverse.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *61–62*

A COMPREHENSIVE FAMILY OF MEMORY KERNELS GENERATED BY THE GAMMA MITTAG-LEFFLER DISTRIBUTION

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Abstract

Systems frequently exhibit history dependence that can be represented through memory kernels. We study the generalized gamma Mittag-Leffler distribution as a unified family of memory kernels encompassing exponential, Erlang, delayed, and heavy-tailed fractional behaviors. This hierarchy establishes explicit connections between ordinary, delay, fractional differential equations, and systems of ordinary differential equations, while preserving a probabilistic interpretation through the kernel moments. The framework has applications in population dynamics, epidemiology, pharmacokinetics, and electrophysiology.

1 Introduction

One of the main sources of memory in mathematical models is the presence of delays. In a kernel-based framework, delay effects can be interpreted as particular cases of distributed memory kernels. Let $y(t)$ denote a system state and suppose that its evolution depends on a memory variable $w(t)$. A broad class of models can be written as $y'(t) = g(t, y(t), w(t))$, where w is defined by the Volterra convolution

$$w(t) = \int_0^t \kappa(t-s) f(s)\, ds; \qquad \kappa(t) \geq 0, \int_0^\infty \kappa(t)\, dt = 1, \tag{1}$$

with f representing the signal whose past contributes to the memory.

Consider the generalized gamma Mittag–Leffler probability density function as the memory kernel:

$$\kappa(x) = \begin{cases} Cx^{\beta-1} e^{-ax} E_{\alpha,\beta}(-\lambda x^\alpha), & x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

This work shows that ordinary, delayed, linear-chain, and fractional differential models arise from different choices of the parameters of the generalized gamma Mittag-Leffler memory kernel.

2 Results

By appropriately selecting the parameters $(a, \beta, \lambda, \alpha)$, one recovers well-known memory kernels:

- **Gamma distribution.** Setting $\lambda = 0$ removes the Mittag-Leffler contribution and yields the gamma probability density function.
- **Erlang distribution (ODE systems through the linear-chain trick).** When $\beta = n \in \mathbb{N}^*$, the gamma distribution reduces to the Erlang distribution.
- **Exponential distribution (classical ODE models).** The particular case $\beta = 1$ of the gamma distribution gives the exponential density.

- **Mittag-Leffler distribution (fractional differential equations).** Setting $a = 0$ and $0 < \alpha = \beta \leq 1$ yields the Mittag-Leffler probability density function, which is characterized by a heavy tail and has no finite mean.

- **Dirac delta distribution (ordinary or delayed differential equations).** The Erlang family converges to a Dirac delta distribution when the mean is kept fixed and the shape parameter tends to infinity. Choosing $a_n = \frac{n}{\mu}$, one obtains $\kappa_n(t) \longrightarrow \delta(t - \mu)$.

Hence, a single probability density function generates the memory kernels underlying several mathematical formulations. This hierarchy provides a continuous framework connecting classical and fractional memory kernels through suitable parameter choices. The parameter β distinguishes weak ($\beta \leq 1$) and strong ($\beta > 1$) kernels, while the Mittag-Leffler contribution may produce either unimodal or multimodal kernels. Consequently, the admissible parameter space naturally partitions into several kernel classes. Complex memory effects can be captured within a coherent formulation, as summarized in Figure 1.

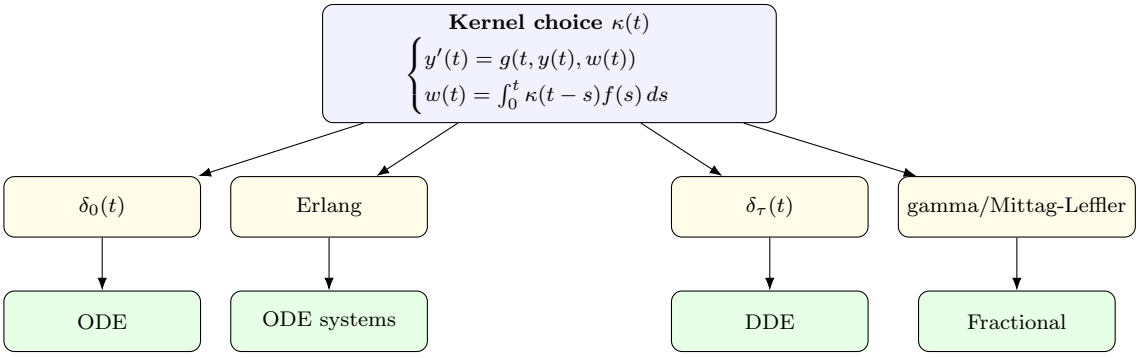


Figure 1: Memory kernels and the corresponding classes of differential models.

Remark. Applications include population dynamics, epidemiology, pharmacokinetics and electrophysiology, demonstrating flexibility across different domains [1, 2, 3, 4].

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XIX ENAMA - Novembro 2026 *63–64*

(ω, C)-PERIODICITY ON ISOLATED TIME SCALES

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Abstract

In this work, we introduce a general concept of (ω, c)-periodicity for functions defined on isolated time scales, which generalizes the concept of ω-periodicity in [5]. We provide important properties such as the equivalence between our definition and an exponential form function, as well as results for existence and uniqueness of that kind of solution for the Volterra integral equation on time scales. Also, we develop necessary and sufficient conditions for the second kind of Volterra integral equation with infinite memory to have a (ω, c)-periodic solution, along with sufficient conditions for a solution of a first order dynamic equation on time scales to be (ω, c)-periodic.

The main objective of this work is to bring the theory of (ω, c)-periodic functions to the time scale environment by generalizing this concept to isolated time scales as an initial investigation. The given definition generalizes the ω-periodicity from [5], along with bringing the anti-periodicity theory and other cases in between for different values of $c \in \mathbb{C} \setminus \{0\}$ to the time scale theory.

Establishing the definition and some initial properties, we give an exponential characterization for this class of functions by writing it as a product between an exponential function and an ω-periodic function. Thus, using this characterization, we define a norm $||\cdot||$ for the space $P_{\omega,c}$ and we show that it is a Banach space under this norm. Moreover, we apply it to discuss the existence and uniqueness of an (ω, c)-periodic solution for linear Volterra integral equation on isolated time scales (of first and second kind). Furthermore, we discuss conditions in order to have (ω, c)-periodic solutions for first order dynamic equations and we illustrate some examples.

1 Main Results

Let $\mathbb{X}$ be a complex Banach space and $\mathbb{T}$ an isolated time scale. For $c \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{N}$, a function $f\colon \mathbb{T} \to \mathbb{X}$ is said to be (ω, c)-periodic if

$$\nu_\omega^\Delta(t) f(\nu_\omega(t)) = c f(t) \text{ for all } t \in \mathbb{T}.$$

The set of all (ω, c)-periodic functions $f\colon \mathbb{T} \to \mathbb{X}$ is denoted by $\mathcal{P}_{\omega,c} = \mathcal{P}_{\omega,c}(\mathbb{T}, \mathbb{X})$.

Next we state the main results.

Theorem 1.1. *$f \in \mathcal{P}_{\omega,c}$ if and only if there exists $p \in P_\omega$ such that*

$$f(t) = c^{\frac{\alpha(t)}{\omega}} p(t) \text{ for all } t \in \mathbb{T}.$$

For $f \in \mathcal{P}_{\omega,c}(\mathbb{T}, \mathbb{X})$, we define the norm $||\cdot||_{\omega,c}$ by

$$||f||_{\omega,c} = \sup_{t \in [t_0, \nu_\omega(t_0)]_\mathbb{T}} ||c^{-\alpha(t)/\omega} \mu(t) f(t)||_\mathbb{X}.$$

Theorem 1.2. *$(\mathcal{P}_{\omega,c}(\mathbb{T},\mathbb{X}),||\cdot||_{\omega,c})$ is a Banach space.*

Theorem 1.3. *Suppose f is L-Lipschitz on the second variable with the property*

$$\nu_\omega^\Delta(t)f(\nu_\omega(t),cx)=cf(t,x) \text{ for all } t\in\mathbb{T},\ x\in\mathbb{X}.$$

Also consider that the integral

$$\int_{-\infty}^{t} a(t,\sigma(s))f(s,\mu(s)x(s))\Delta s$$

exists for each $t\in\mathbb{T}$, $a\colon \mathbb{T}\times\mathbb{T}\to\mathbb{C}$ satisfies

$$\int_{-\infty}^{u}|a(t,\sigma(s))|\Delta s\leq\int_{-\infty}^{u}Ke_{\ominus\gamma}(t,\sigma(s))\Delta s \text{ for every } t,u\in\mathbb{T}$$

for some constants $K,\gamma\in\mathbb{R}^+$ and

$$a(\nu_\omega(t),\nu_\omega(s))=a(t,s),$$

for each $t,s\in\mathbb{T}$ and $\frac{2KL}{\gamma}(1+\tilde{\mu}\gamma)<1$ where $\tilde{\mu}=\sup_{t\in[t_0,\nu_\omega(t_0))_{\mathbb{T}}}\mu(t)$.
Then equation

$$x(t)=\frac{1}{\mu(t)}\int_{-\infty}^{t}a(t,\sigma(s))f(s,\mu(s)x(s))\Delta s$$

has a unique (ω,c)-periodic solution.

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XIX ENAMA - Novembro 2026 *65–66*

CRITICAL DOUBLE PHASE EQUATIONS IN $\mathbb{R}^N$ WITH LOGARITHMIC NONLINEARITIES: WHAT'S NEW?

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Abstract

In this talk, we discuss about double phase equations set in $\mathbb{R}^N$ involving critical Sobolev nonlinearities and logarithmic perturbations. More precisely, our equations are driven by the so-called *double phase operator* given by

$$u \in W^{1,\mathcal{H}}(\mathbb{R}^N) \mapsto \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{p-2}\nabla u\right),$$

set on an appropriate Musielak-Orlicz-Sobolev space $W^{1,\mathcal{H}}(\mathbb{R}^N)$, with $1 < p < q < \infty$ and $\mu \in L^\infty(\mathbb{R}^N)$ such that $\mu(x) \geq 0$ a.e. in $\mathbb{R}^N$.

The main difficulties arise from the presence of the logarithmic perturbation, which is sign-changing, combined with a double lack of compactness, due to the free action of translation group in $\mathbb{R}^N$ and the critical Sobolev nonlinearity. Furthermore, we have to deal with the Luxemburg type norm of $W^{1,\mathcal{H}}(\mathbb{R}^N)$, which complicates even the study of geometry for the energy functional. For this, we introduce two different strategies to get existence results for our equations, and we compare the pros and cons.

Our results are new even in the classical p-Laplacian case, that is when $\mu \equiv 0$.

1 Introduction

In this talk, we introduce the following equation in $\mathbb{R}^N$

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + \mu(x)|u|^{q-2}u = K_1(x)|u|^{p^*-2}u + \lambda K_2(x)|u|^{r-2}u\log(|u|) \\ + \gamma K_3(x)|u|^{\beta-2}u, \end{aligned} \tag{1}$$

where the main operator on the left-hand side is the so-called double phase operator satisfying the structural assumption:

(H_1) $1 < p < q < N$, $q < p^* = \frac{Np}{N-p}$ *and* $\mu\colon \mathbb{R}^N \to [0,\infty)$ *is Lipschitz continuous such that* $\mu \in L^\infty(\mathbb{R}^N)$.

Here, we consider parameters λ, $\gamma > 0$ and exponents $r \in [q, p^*)$, $\beta \in (1, p^*)$. Concerning the functions K_1, K_2, $K_3 : \mathbb{R}^N \to \mathbb{R}$, we assume the following standard conditions:

(H_2) $K_1 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $K_1(x) > 0$ *for all* $x \in \mathbb{R}^N$ *and if* $\{A_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ *is a sequence of Borel sets such that the Lebesgue measure* $|A_n| \leq R$ *for all* $n \in \mathbb{N}$ *and some* $R > 0$, *then*

$$\lim_{n\to\infty} \int_{A_n \cap B_\rho^c(0)} K_1(x)\,\mathrm{d}x = 0,$$

for some $\rho > 0$.

(H_3) $K_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *with* $0 < K_2 < K_1$ *on* $\mathbb{R}^N$, *and* $K_3 \in L^\infty(\mathbb{R}^N)$ *with* $0 < K_3 < K_1$ *on* $\mathbb{R}^N$.

2 Main Results

During the XVII ENAMA which occurred in Rio de Janeiro in 2024, we provided the following existence result for (1), proved in [1].

Theorem 2.1. *Let $\gamma = \lambda$. Let (H_1) – (H_3) be satisfied and let $1 < p < q < \beta < r < p^*$. Furthermore, assume that:*

(H_4) *there exists $\sigma \in (q, \beta)$ such that*

$$K_2(x) \leq \frac{e\,(r-\beta)\,r\,(\beta-\sigma)}{\beta(r-\sigma)} K_3(x), \quad \text{for any } x \in \mathbb{R}^N.$$

Then, there exists $\lambda^* > 0$ such that, if $\lambda \geq \lambda^*$, equation (1) admits at least one nontrivial mountain pass weak solution.

The proof of Theorem 2.1 strongly relies on the presence of a β-exponent nonlinearity in (1). In order to avoid this presence, by considering $\gamma = 0$ in (1), we introduce a new strategy. The next result is proved in [2].

Theorem 2.2. *Let $\gamma = 0$. Let (H_1) – (H_3), let $1 < p < q$ and $\max\left\{p, p^* - \frac{p}{p-1}\right\} < r < p^*$. Furthermore, assume that:*

(H_5) *there exists $R > 0$ such that $\mu(x) = 0$, $K_1(x) = \|K_1\|_\infty$ and $K_2(x) = \|K_2\|_\infty$ for a.e. $x \in B_R(0)$.*

Then, for any $\lambda > 0$ equation (1) admits at least one nontrivial weak solution.

Aim of the present talk is to compare the two strategies used in Theorems 2.1 and 2.2. In particular, we emphasize the different assumptions on parameters λ and γ, on exponent r, on functions μ, K_1, K_2 and K_3. Furthermore, even if we exploit a mountain pass argument in both theorems, the nontrivial weak solution of Theorem 2.2 is not of mountain pass type.

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XIX ENAMA - Novembro 2026 *67–68*

EXISTENCE OF SOLUTIONS FOR A SEMILINEAR ELLIPTIC PROBLEM VIA DIRECT METHOD AND THE MOUNTAIN PASS THEOREM

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Abstract

In this work, we study the existence of weak solutions for a class of semilinear elliptic Dirichlet problems by means of variational methods. More precisely, we consider

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain and $V \in L^\infty(\Omega)$ satisfies suitable assumptions. The problem is associated with an energy functional defined on the Sobolev space $H_0^1(\Omega)$, whose critical points correspond to weak solutions of the equation. We investigate two distinct regimes for the nonlinear term: the sublinear case, when $1 < p < 2$, and the superlinear subcritical case, when $2 < p < 2^*$. According to the growth of the nonlinear term, we use the Direct Method of the Calculus of Variations and the Mountain Pass Theorem as tools.

1 Introduction

Variational methods provide an important framework for studying the existence of solutions to nonlinear partial differential equations. In this approach, a differential equation is associated with an energy functional defined on a suitable space, and weak solutions are obtained as critical points of this functional.

We associate with the semilinear elliptic Dirichlet problem (1) the energy functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega V(x)u^2\,dx - \int_\Omega F(x,u)\,dx, \tag{2}$$

where

$$F(x,u) = \int_0^u f(x,s)\,ds.$$

In the sublinear case $1 < p < 2$, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a CarathÃ©odory function satisfying appropriate growth conditions. Under these hypotheses, the associated energy functional is well-defined, continuously differentiable, coercive, and weakly lower semicontinuous. Consequently, the Direct Method of the Calculus of Variations can be employed to analyze the existence of weak solutions.

In the superlinear subcritical case $2 < p < 2^*$, we assume that f satisfies a subcritical growth condition and a suitable superlinearity hypothesis, such as the Ambrosetti–Rabinowitz condition. In this setting, we investigate the geometric structure of the associated functional and verify the compactness properties required for the application of minimax techniques. In particular, the Mountain Pass Theorem provides the appropriate variational framework for studying nontrivial critical points.

These results illustrate how different growth behaviors of the nonlinearity lead to distinct variational structures and require complementary analytical techniques for the study of semilinear elliptic equations.

2 Main Results

The following are the main results established throughout this work.

Theorem 2.1. *Assume that $V \in L^\infty(\Omega)$ satisfies $V(x) \geq V_0 > -\lambda_1(\Omega)$ for a.e. $x \in \Omega$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$|f(x,u)| \leq a(x)|u| + b(x)|u|^{p-1}, \quad 1 < p < 2,$$

with $a, b : \Omega \to \mathbb{R}$ satisfying

$$V_0 < \|a\|_{L^\infty} < V_0 + \lambda_1 \text{ and } b \in L^\infty(\Omega).$$

Then the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega V(x)u^2\,dx - \int_\Omega F(x,u)\,dx$$

is weakly lower semicontinuous, coercive, and of class C^1 on $H_0^1(\Omega)$.

Therefore, by the Direct Method of the Calculus of Variations, the energy functional J possesses a global minimizer, which yields a weak solution of problem (1). Under suitable additional assumptions, this solution is nontrivial.

Theorem 2.2. *Assume that $V \in L^\infty(\Omega)$ satisfies $V(x) \geq V_0 > -\lambda_1(\Omega)$ for a.e. $x \in \Omega$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a CarathÃ©odory function satisfying*

(i) $|f(x,s)| \leq C(1 + |s|^{p-1})$, *for some* $2 < p < 2^*$

(ii) $\lim\limits_{t\to 0} \dfrac{f(x,t)}{t} = 0$;

(iii) $0 < \mu F(x,s) \leq sf(x,s)$, *for some* $\mu > 2$ *and sufficiently large* $|s|$.

Then the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega V(x)u^2\,dx - \int_\Omega F(x,u)\,dx$$

is of class C^1, satisfies the mountain pass geometry and satisfies the Palais–Smale condition.

Therefore, by the Mountain Pass Theorem, the energy functional J possesses a nontrivial critical point, which corresponds to a weak solution of problem (1).

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XIX ENAMA - Novembro 2026 *69–70*

ON A SPECIAL CLASS OF NONLOCAL NEUMANN PROBLEMS WITH CRITICAL GROWTH

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Abstract

In this work, we investigate existence and multiplicity of nontrivial solutions for a nonlocal Neumann problem, with a particular focus on the case where the nonlocal term exhibits a decreasing behavior. Our main results are derived by utilizing the concentration-compactness principle originally introduced by Lions and extended by Bonder, Saintier and Silva. By employing trace theory and the Mountain Pass Theorem, we demonstrate that, under suitable conditions, the problem admits a nontrivial solution, and multiplicity of solutions is obtained via a truncation argument.

1 Introduction

In this article, we study via variational method, existence and multiplicity of nontrivial solutions for the following nonlocal Neumann problem

$$\begin{cases} \mathbb{L}(u) + \vartheta|u|^{h(x)-2}u = \lambda|u|^{z(x)-2}u + g(x,u) & \text{in} \quad \Omega, \\ \mathbb{L}_\nu(u) = \gamma|u|^{r(x)-2}u\left[\displaystyle\int_{\partial\Omega}\frac{1}{r(x)}|u|^{r(x)}\,dS\right]^{\beta} & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$,

$$\mathbb{L}(u) = M(\psi(u))\left(-\Delta_{p(x)}u + |u|^{p(x)-2}u\right), \quad \mathbb{L}_\nu(u) = M(\psi(u))|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial \nu},$$

$p \in C(\bar{\Omega})$ with $1 < p(x) < N$, $r \in C(\partial\Omega)$, $h, z \in C(\Omega)$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $\vartheta, \lambda, \gamma, \beta$ are positive parameters, and $\Delta_{p(x)}$ is the $p(x)-$Laplace operator.

The function $M : [0, +\infty) \to \mathbb{R}$ is given by

$$M(\tau) = a - b\tau^{\eta}, \tag{2}$$

with $a > 0, b > 0$, $\eta \geq 1$ and $\psi(u) = \displaystyle\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)\,dx$.

For the nonlinearity term $g(x,t) \in C(\bar{\Omega} \times \mathbb{R})$, we consider the same assumptions:

(A1) the subcritical growth condition:

$$|g(x,s)| \leq C(1 + |s|^{q(x)-1}),$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where $q \in C(\bar{\Omega})$, $C > 0$ and $z(x) < q(x) < p_*(x)$;

(A$'$1) the subcritical growth condition:

$$|g(x,s)| \leq C(1 + |s|^{q(x)-1}),$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where $q \in C(\bar{\Omega})$, $C > 0$ and $p(x) < q(x) < p^*(x)$;

(A2) $\lim_{s\to 0} \frac{g(x,s)}{|s|^{z(x)-2}s} = 0$;

(A$'$2) $\lim_{s\to 0} \frac{g(x,s)}{|s|^{p(x)-2}s} = 0$;

(A3) there exists $\theta \in \left(p^+, \min\left\{z^-, \frac{(\eta+1)(p^-)^{\eta+1}}{(p^+)^{\eta}}, \frac{(\beta+1)(r^-)^{\beta+1}}{(r^+)^{\beta}}\right\}\right)$ such that

$$0 < \theta G(x,s) \leq sg(x,s), \text{for all } s > 0, x \in \Omega,$$

where

$$G(x,s) = \int_0^s g(x,t)dt.$$

(A$'$3) there exists $\theta \in \left(h^+, \min\left\{\frac{(\eta+1)(p^-)^{\eta+1}}{(p^+)^{\eta}}, \frac{(\beta+1)(r^-)^{\beta+1}}{(r^+)^{\beta}}\right\}\right)$ such that

$$0 < \theta G(x,s) \leq sg(x,s), \quad \text{for all } s > 0,\ x \in \Omega,$$

where

$$G(x,s) = \int_0^s g(x,t)dt.$$

This hypothesis is known as the Ambrosetti-Rabinowitz's condition.

(A4) $g(x,-s) = -g(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$.

2 Main Results

The main results are the following:

Proposition 2.1. *Assume (2), $\vartheta = 0$, $p, z, q \in \mathcal{C}^+(\bar{\Omega})$, $r \in \mathcal{C}^+(\partial\Omega)$ (see Section 2) satisfying*

$$1 < p^- \leq p(x) \leq p^+ < z^- \leq z(x) \leq z^+ < q^- \leq q(x) \leq q^+ < r^- \leq r(x) \leq p_*(x),$$

and (A1)-(A3). Moreover, assume $z^- < p^-(\eta+1) < r^-(\beta+1)$. Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ and for all $\gamma > 0$, the problem (1) has a nontrivial solution in $W^{1,p(x)}(\Omega)$.

Proposition 2.2. *Assume (2), $\vartheta = 1$, $p, h, q \in C^+(\bar{\Omega})$, $r \in C^+(\partial\Omega)$ (see Section 2) satisfying*

$$1 < r^- \leq r(x) \leq r^+ \leq p_*(x) < p^- \leq p(x) \leq p^+ < h^- \leq h(x) \leq h^+ < q^- \leq q(x) \leq q^+ < p^*(x),$$

(A′1)-(A′3) and (A4). Moreover, assume $h^+ < r^-(\beta+1)$ and $1 < z^- \leq z(x) \leq z^+ < p^-$. Then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ and for all $\gamma > 0$, there exist infinitely many solutions to the problem (1) in $W^{1,p(x)}(\Omega)$.

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XIX ENAMA - Novembro 2026 *71–72*

ON THE EXISTENCE, REGULARIZING EFFECTS, AND ASYMPTOTIC BEHAVIOR OF (P, Q)-LAPLACIAN SCHRÖDINGER-MAXWELL SYSTEMS

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Abstract

We study a class of coupled quasilinear Schrödinger-Maxwell type systems driven by mixed (p, q)-Laplacian operators. We establish existence and regularity results for solutions under low summability assumptions on the source term, overcoming difficulties caused by the lack of a maximum principle and standard variational structures.

Our approach is based on a suitably perturbed auxiliary problem, which yields the required a priori estimates and compactness properties. A further difficulty arises from the nonlinear coupling, which prevents the direct use of the classical composition of solution operators. To handle this issue, we develop a fixed point framework adapted to the structure of the system.

As a consequence, we obtain existence results in both regular and non-regular regimes and identify ranges where solutions exhibit a regularizing effect, including improved Sobolev and Lebesgue regularity. We also investigate a higher regularity regime and analyze the asymptotic behavior of solutions as $p \to \infty$, proving convergence to Lipschitz continuous limit profiles. Our results extend previous works and show that the regularizing effect persists in the presence of mixed operators and strong nonlinear coupling.

1 Introduction

More recently, attention has shifted toward the study of Schrödinger-Maxwell type systems. These models introduce additional analytical difficulties, since the operators are generally not self-adjoint and classical tools, such as comparison principles and duality arguments, are no longer directly applicable. Nevertheless, it has been shown that the regularizing effect persists even in this more general framework; see, for instance, [1, 2].

In this paper, we study a Schrödinger-Maxwell type system driven by the (p, q)-Laplacian operator. Our main objective is to establish existence and regularity results for weak solutions, as well as to identify parameter regimes in which regularizing effects occur, particularly in the case $\theta > 0$. The combination of mixed quasilinear operators and highly nonlinear coupling terms gives rise to significant analytical challenges. In particular, the (p, q)-structure prevents the direct application of several standard techniques available for homogeneous operators, while the nonlinear coupling makes the construction of suitable solution operators considerably more delicate.

More precisely, we consider the following system:

$$\begin{cases} -\Delta_p w - \mu\Delta_q w + |w|^{r-2}w|z|^{\theta+1} = f & \text{in } \Omega, \\ -\Delta_p z - \mu\Delta_q z = |w|^r|z|^\theta & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, $f \in L^t(\Omega)$ for some $t > 1$, satisfying $f > 0$ almost everywhere in Ω, $\mu \geq 0$, $r > 1$, $0 < \theta < p - 1$, and $1 < q < p < \infty$.

2 Main Results

Theorem 2.1. *Let* $0 < \theta < \min\left\{\frac{p^2}{N-p}, p-1\right\}, 1 < q < p < N$, *and let* $f \in L^t(\Omega)$ *be such that* $f > 0$ *a.e. in* Ω. *Then the following statements hold.*

(i) Assume that $1 < r < p^*$ *and* $(p^*)' \leq t < \frac{N}{p}$. *Then there exists a solution* $(w, z) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *to problem* (1) *satisfying* $w \geq 0$ *a.e. in* $\Omega, z \geq 0$ *a.e. in* Ω.

(ii) Assume that $\theta < q-1, r > 1$ *and* $t \geq (r+\theta+1)'$. *Then there exists a solution* $(w, z) \in \left(W_0^{1,p}(\Omega) \cap L^{r+\theta+1}(\Omega)\right) \times W_0^{1,p}(\Omega)$ *to problem* (1) *satisfying* $w \geq 0$ *a.e. in* $\Omega, z \geq 0$ *a.e. in* Ω.

(iii) Assume that $r > 1$ *and* $t > \frac{N}{p}$. *Then there exists a bounded regularized solution* $(w, z) \in \left(W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\right)^2$ *to problem* (1) *satisfying* $w \geq 0$ *a.e. in* Ω, $z \geq 0$ *a.e. in* Ω.

Theorem 2.2. *Let* $p > N$ *and let* $f \in L^t(\Omega)$ *be such that* $f > 0$ *a.e. in* Ω, *with* $t > 1$. *Assume that* $1 < q(p) < p$ *and* $\lim_{p\to+\infty} q(p) = +\infty$. *Then there exists* $(w_p, z_p) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *satisfying*

$$\begin{cases} -\Delta_p w_p - \mu\Delta_{q(p)} w_p + |w_p|^{r-2} w_p |z_p|^{\theta+1} = f & \text{in } \Omega, \\ -\Delta_p z_p - \mu\Delta_{q(p)} z_p = |w_p|^r |z_p|^\theta & \text{in } \Omega, \\ w_p = z_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Moreover, $w_p \geq 0$ *and* $z_p \geq 0$ *a.e. in* Ω. *Furthermore, up to subsequences,* $w_p \to w_\infty$ *and* $z_p \to z_\infty$ *uniformly in* $\overline{\Omega}$ *as* $p \to +\infty$, *for some* w_∞, $z_\infty \in C_0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. *In addition,* $0 \leq w_\infty(x) \leq d_\Omega(x)$, $0 \leq z_\infty(x) \leq d_\Omega(x)$ *for all* $x \in \overline{\Omega}$, *and* $\|\nabla w_\infty\|_{L^\infty(\Omega)} \leq 1$, $\|\nabla z_\infty\|_{L^\infty(\Omega)} \leq 1$. *Finally,*

$$z_\infty = d_\Omega \quad \text{in } \overline{\{x \in \Omega : w_\infty(x) > 0\}}.$$

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XIX ENAMA - Novembro 2026 *73–74*

FITZHUGH-NAGUMO-TYPE SYSTEMS WITH SIGN-CHANGING COEFFICIENTS

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Abstract

We establish existence and multiplicity results for steady-state solutions of spatially heterogeneous FitzHugh–Nagumo-type systems in Euclidean space, extending the existing theory from constant to variable coefficients that may change sign. We prove the existence of three distinct weak solutions: one componentwise positive, one componentwise negative, and one sign-changing. The proof combines variational methods with Weth's theorem, whose use in this context is new. A key ingredient is the construction of a suitable Hilbert space, with a carefully chosen scalar product, which ensures that the associated polar cone satisfies the assumptions required for the application of Weth's theorem.

1 Problem and variational setting

The FitzHugh–Nagumo model describes the interaction between an activator and an inhibitor in excitable media. Most variational studies of its steady states assume constant positive coefficients [1, 2]. We consider instead

$$\begin{cases} -\Delta u + a(x)v &= f(x,u) \quad \text{in } \mathbb{R}^N, \\ -\Delta v + b(x)v &= \beta a(x)u \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 3$, $\beta > 0$, and $a, b \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ are bounded below. They are not required to be coercive or to decay at infinity.

We work with classes $\Upsilon_s(\mathbb{R}^N)$ of coefficients σ for which the quadratic form is positive,

$$\lambda_1(\sigma) = \inf_{\substack{\varphi \in C_0^\infty(\mathbb{R}^N) \\ \|\varphi\|_2 = 1}} \int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 + \sigma(x)\varphi^2 \right) dx > 0,$$

and suitable local Rayleigh quotients tend to infinity on balls whose centers escape to infinity. This condition yields compactness without imposing coercivity and admits coefficients taking negative values. The natural space is

$$E_b = \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} b^+(x) u^2 \, dx < \infty \right\}.$$

The second equation in (1) can be solved in E_b. With $\mathcal{L}u = \beta(-\Delta + b(x))^{-1}(a(x)u)$, the system is equivalent to

$$-\Delta u + a(x)\mathcal{L}u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{2}$$

If the energy spaces generated by a and b coincide with equivalent norms, then

$$\langle u, w \rangle_{a,b} = \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla w + a(x) u \, \mathcal{L} w \right) dx$$

is a scalar product and $\|u\|^2_{a,b} = \|\nabla u\|_2^2 + \beta^{-1}\|\mathcal{L}u\|_b^2$. The embeddings of E_b into the relevant subcritical Lebesgue spaces are compact. Hence the energy

$$J(u) = \frac{1}{2}\|u\|^2_{a,b} - \int_{\mathbb{R}^N} F(x,u)\, dx, \qquad F(x,u) = \int_0^u f(x,t)\, dt,$$

has the compactness properties required by minimax methods.

2 Main results

For the power nonlinearity, a representative result is the following.

Theorem 2.1. *Let $b \in \Upsilon_2(\mathbb{R}^N)$, assume that the energy spaces associated with a and b coincide with equivalent norms, and let $1 < p < 2^* - 1$, where $2^* = 2N/(N-2)$. Then*

$$\begin{cases} -\Delta u + a(x)v &= |u|^{p-1}u, \\ -\Delta v + b(x)v &= a(x)u \end{cases} \qquad \text{in } \mathbb{R}^N$$

has a nontrivial mountain pass solution. If, in addition, $a > 0$ almost everywhere and $b - 2a$ satisfies the structural assumption defining the modified energy norm, then there exist three nontrivial weak solutions (u_i, v_i) such that

$$u_1 > 0, \quad v_1 > 0; \qquad u_2 < 0, \quad v_2 < 0; \qquad u_3 \text{ changes sign.}$$

The central point of the multiplicity result is the use of Weth's theorem [3], based on a positive cone and its polar cone. Its application in the present nonlocal setting is not direct. The original scalar product does not provide the maximum principle needed to control the polar cone. We overcome this obstruction by adding a suitable linear term to both sides of (2) and introducing an equivalent scalar product. In the modified Hilbert structure, a maximum principle for the nonlocal operator implies the decisive inclusion of the polar cone into the negative cone. Weth's theorem then yields a positive, a negative, and a sign-changing critical point.

The mountain-pass argument gives existence under the more general sign-changing assumptions, while the cone construction gives the three solutions under the additional positivity and monotonicity hypotheses. The arguments extend to Carathéodory nonlinearities with weighted subcritical growth and provide a framework for other elliptic systems with heterogeneous, non-coercive coefficients.

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XIX ENAMA - Novembro 2026 *75–76*

FRACTIONAL HARDY-HÉNON EQUATIONS WITH NONLINEARITIES OF CRITICAL EXPONENTIAL GROWTH

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Abstract

In this work, we study a class of fractional Hardy-Hénon problems with nonlinearities of critical exponential growth in the sense of the Trudinger-Moser inequality. By means of a suitable change of variables, we establish a weighted Trudinger-Moser inequality in the radial setting, which yields a critical exponent depending on the weight parameter α. This framework encompasses both Hénon-type ($\alpha > 0$) and Hardy-type ($-1 < \alpha < 0$) equations.

1 Introduction

We investigate the following class of fractional Hardy–Hénon problems:

$$\begin{cases} (-\Delta)^{1/2}u &= \lambda|x|^{\mu}u + |x|^{\alpha}f(u) \quad \text{in} \quad (-1,1), \\ u &= 0 \qquad\qquad\qquad\quad \text{on} \quad \mathbb{R}\setminus(-1,1), \end{cases} \tag{1}$$

where $\mu \geq \alpha > -1$, and the nonlinearity f exhibits critical exponential growth in the sense of the Trudinger–Moser inequality.

Our approach is based on a suitable change of variables that exploits the behavior of the weight $|x|^{\alpha}$. This transformation yields a weighted Trudinger–Moser inequality in the radial setting, allowing us to study the problem within a nonlocal variational framework and to identify a critical exponent that depends explicitly on the parameter α. When $\alpha > 0$, the problem becomes of Hénon type, and the corresponding critical exponent is larger than the classical one. This phenomenon can be viewed as the fractional counterpart of the result of Ni [3] for the local Laplacian in $\mathbb{R}^N$ ($N \geq 3$). On the other hand, when $-1 < \alpha < 0$, the problem is of Hardy type. In this case, the critical exponent is smaller than the classical one, while the singularity of the weight at the origin introduces additional analytical difficulties.

The main challenge is to overcome the lack of compactness associated with nonlinearities of critical exponential growth. To this end, we employ variational methods and estimate the minimax level by means of suitable Moser-type functions (see [4]). Under appropriate assumptions on the parameters λ, μ, and α, as well as on the nonlinearity f, we prove the existence of at least one nontrivial radial solution. Depending on the position of λ with respect to the spectrum of the fractional Laplacian, our existence results are obtained via either the Mountain Pass Theorem or the Linking Theorem. Consequently, our results provide a fractional counterpart of those in [1] and extend the existence theory established in [2].

2 Main Results

Before stating our main results, we introduce some notation and hypotheses.

We say that the nonlinearity f has *critical exponential growth* at $+\infty$ if

$$(CG) \qquad \lim_{t\to+\infty} \frac{|f(t)|}{e^{\tau t^2}} = \begin{cases} 0, & \forall \tau > \pi(1+\alpha), \\ +\infty, & \forall \tau < \pi(1+\alpha). \end{cases}$$

Here (λ_j^*) denotes the sequence of eigenvalues associated with the radial fractional Laplacian operator. We assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses.

(f_1) (Behavior at zero) $f : \mathbb{R} \to \mathbb{R}^+$ is continuous with $f = 0$ on $\mathbb{R}^-$ and for each $j \geq 1$,

$$\limsup_{s\to 0^+} \frac{2f(s)}{r^{\beta(\mu+1)-1}s} = l \text{ with } \beta^2 l < \lambda_j^* - \lambda, \text{uniformly for } r \in (0,1),$$

where $\lambda \in (0, \lambda_1^*)$ if $j = 1$, otherwise $\lambda \in (\lambda_{j-1}^*, \lambda_j^*)$, also $l = 0$ if $\lambda = \lambda_j^*$,

(f_2) There exist $M > 0$ and $s_0 > 0$ such that $0 < F(s) \leq Mf(s)$ for all $s > s_0$, where $F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau$.

(f_3) $0 < 2F(s) \leq f(s)s$ for all $s > 0$.

(f_4) $\lim\limits_{s\to+\infty} \dfrac{f(s)s}{\exp(\pi(1+\alpha)s^2)} \geq \xi$ with $\xi > \dfrac{(1+\alpha)}{2\widehat{M}}$, where

$$M_\epsilon = \lim_{n\to\infty} \int_0^1 ne^{n((1-\epsilon)t^2-t)}\, dt \quad \text{and} \quad \widehat{M} = \lim_{\epsilon\to 0} M_\epsilon.$$

We denote by M_0 the quantity M_ε corresponding to $\varepsilon = 0$. A straightforward computation shows that

$$M_0 = 2 \quad \text{and} \quad \widehat{M} = 1.$$

Theorem 2.1. *In addition to (CG) and $(f_1 - f_4)$, suppose $\alpha > -1$ and the following condition holds, $\lambda_j^* \leq \lambda < \lambda_{j+1}^*$, $j \geq 0$, and we denote $\lambda_0^* = 0$. Then Problem (1) possesses a nontrivial radially symmetric solution.*

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XIX ENAMA - Novembro 2026 *77–78*

A CONVECTIVE 1-HOMOGENEOUS P-LAPLACIAN PROBLEM

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Abstract

We study a two-parameter positively 1-homogeneous p-Laplacian problem involving a convection term.

1 Introduction

We study the following two-parameter Dirichlet problem involving a convection term

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + \mu \left|\nabla u\right|^{p-1} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, the parameters λ and μ are nonnegative, and $p > 1$.

This problem is positively 1-homogeneous: if u is a weak solution to (1), then ku is also a weak solution for any positive constant k.

There are few existence results in the literature related to 1-homogeneous Dirichlet problems involving the p-Laplacian and a convection term. One of them is obtained by Shouchuan and Papageorgiou in [4] for the problem (1), with μ replaced with $a(x)$, a continuous function of x that may change sign. They used the theory of nonlinear operators of monotone type to prove that if

$$\|a\|_\infty < \lambda_{1,p}(\Omega)^{1/p}, \tag{2}$$

then the problem admits an L^∞-normalized weak solution for some positive constant λ. Here, $\lambda_{1,p}(\Omega)$ denotes the first eigenvalue of the Dirichlet p-Laplacian.

We first show that the restriction (2) on $\|a\|_\infty$ can be dropped when the domain is the ball $B_R := \left\{x \in \mathbb{R}^N : |x| \leq R\right\}$, or when Ω is a general bounded domain and a is a nonnegative function.

Then, we use the results related to the ball to prove that for each $\mu > 0$ there exists a pair (λ_μ, u_μ) such that $\lambda_\mu > 0$ and u_μ is a weak solution to (1) with $\lambda = \lambda_\mu$. Further, we estimate λ_μ and prove that $\mu \mapsto \lambda_\mu$ is a nonincreasing function.

Our results are stated in the next section.

2 Main Results

Theorem 2.1. *Let $a \in C(\overline{B_R})$. The problem*

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + a(x) \left|\nabla u\right|^{p-1} & \text{in} \quad B_R \\ u > 0 & \text{in} \quad B_R \\ u = 0 & \text{on} \quad \partial B_R, \end{cases}$$

admits a weak solution $u \in C^1(\overline{B_R}) \cap W_0^{1,p}(B_R)$, for some $\lambda > 0$ that can be estimated as follows

$$0 < \frac{N}{R^p} \frac{\min\left\{1, e^{\max_{|x|\leq R} a(x)}\right\}}{\max\left\{1, e^{\max_{|x|\leq R} a(x)}\right\}} \leq \lambda < \lambda_{1,p}(B_R).$$

Theorem 2.2. *Let $a \in C(\overline{\Omega})$ be nonnegative and let $R > 0$ be such that $\Omega \subset B_R$. The problem*

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + a(x)\,|\nabla u|^{p-1} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

admits a weak solution $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$, for some $\lambda > 0$ that can be estimated as follows

$$0 < \frac{N}{R^p} e^{-\|a\|_\infty R} \leq \lambda < \lambda_{1,p}(\Omega).$$

Theorem 2.3. *Let $\mu > 0$ be fixed and let $R > 0$ be such that $\Omega \subset B_R$. There exists a pair (λ_μ, u_μ) such that $\lambda_\mu > 0$ and $u_\mu \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is a weak solution to*

$$\begin{cases} -\Delta_p u = \lambda_\mu u^{p-1} + \mu\,|\nabla u|^{p-1} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover,

$$N\frac{e^{-\mu_2 R}}{R^p} < \lambda_{\mu_2} \leq \lambda_{\mu_1} < \lambda_{1,p}(\Omega) \text{ whenever } 0 < \mu_1 < \mu_2. \tag{1}$$

The idea behind the proof of Theorem 2.3 is inspired by [1], in which Biezuner, Brown, Ercole, and Martins showed that the first eigenvalue-eigenfunction pair of the p-Laplacian can be obtained via a limit process, where compactness follows from a well-known global regularity result by Lieberman [3]. To deal with the convective term we apply results of Cuesta Leon [2], and to guarantee the monotonicity stated in (1) we use Helly's selection principle.

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XIX ENAMA - Novembro 2026 *79–80*

EXISTENCE AND CONCENTRATION SOLUTIONS FOR A COUPLED ELLIPTIC SYSTEMS WITH CRITICAL EXPONENTIAL GROWTH IN $\mathbb{R}^2$ AND POTENTIALS WELL

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Abstract

We study a gradient system in $\mathbb{R}^2$ given by

$$\begin{cases} -\Delta u + [1 + \mu W_1(x)]\, u = Q_u(u,v) & \text{in } \mathbb{R}^2, \\ -\Delta v + [1 + \mu W_2(x)]\, v = Q_v(u,v) & \text{in } \mathbb{R}^2. \end{cases}$$

The nonlinearity Q has exponential subcritical or critical growth. We prove the existence of a positive weak solution with minimal energy $(u_\mu, v_\mu) \in L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2)$, for some $0 < \iota < 1$. We also show a concentration result for this positive solution (u_μ, v_μ) as $\mu \to +\infty$.

1 Introduction

We study a gradient system in $\mathbb{R}^2$ given by

$$\begin{cases} -\Delta u + [1 + \mu W_1(x)]\, u = Q_u(u,v) & \text{in } \mathbb{R}^2, \\ -\Delta v + [1 + \mu W_2(x)]\, v = Q_v(u,v) & \text{in } \mathbb{R}^2. \end{cases}$$

The nonlinearity Q has exponential subcritical or critical growth. We prove the existence of a positive weak solution with minimal energy $(u_\mu, v_\mu) \in L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2) \cap C^{1,\iota}_{loc}(\mathbb{R}^2)$, for some $0 < \iota < 1$. We also show a concentration result for this positive solution (u_μ, v_μ) as $\mu \to +\infty$. The potentials satisfy the following hypotheses:

(W_1) The potentials are non-negative, i.e., for $i \in \{1,2\}$,

$$W_i(x) \geq 0 \text{ for all } x \in \mathbb{R}^2.$$

(W_2) The sets

$$\Omega_1 := \operatorname{int}\left\{x \in \mathbb{R}^2;\ W_1(x) = 0\right\} \quad \text{and} \quad \Omega_2 := \operatorname{int}\left\{x \in \mathbb{R}^2;\ W_2(x) = 0\right\}$$

are non-empty. Moreover, $\Omega_0 = \Omega_1 \cap \Omega_2$ is a non-empty bounded domain with smooth boundary $\partial\Omega_0$.

(W_3) There exist $W_1^* > 0$ and $W_2^* > 0$, such that

$$meas\left(\left\{x \in \mathbb{R}^2 : W_i(x) \leq W_i^*\right\}\right) < \infty,\ i \in \{1,2\}.$$

Our interest in this system was motivated by the study made by Bartsch and Wang [1] in the equation

$$-\Delta u + [1 + \mu V(x)]\, u = f(x,u) \ \text{ in } \ \mathbb{R}^N, \quad N \geq 3, \tag{BW_μ}$$

where f had a behavior as a power $f(x,t) = |t|^{p-2}t$, $2 < p < 2^* = \frac{2N}{N-2}$. They proved that (BW_μ) has a positive ground state solution u_μ for μ large and u_μ converges strongly for u in $H^1(\mathbb{R}^N)$, as $\mu \to +\infty$, where u is a ground state solution of the limiting equation

$$-\Delta u + u = f(u), \quad \text{in } \Omega_0,$$

where $\Omega_0 = V^{-1}(0)$. This class of potential V became known in the literature as steep potential well. The version of [1] with the multiplicity result was showed by the same authors in [2].

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XIX ENAMA - Novembro 2026 *81–82*

SYMMETRY AND CLASSIFICATION OF POSITIVE STANDING WAVES OF NONLINEAR HARTREE TYPE EQUATIONS

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Abstract

This work is concerned with qualitative properties and the classification of positive solutions to the following system modeling standing waves of coupled Hartree-type equations:

$$\begin{cases} -\Delta u + \tau u = \frac{2p}{p+q}\left(I_\alpha * |v|^q\right)|u|^{p-2}u; \\ -\Delta v + \eta v = \frac{2q}{p+q}\left(I_\alpha * |u|^p\right)|v|^{q-2}v \end{cases} \quad \text{in} \quad \mathbb{R}^N,$$

where $N \in \mathbb{N}$; $0 < \alpha < N$; $\tau, \eta > 0$ and $p, q > 1$. By means of the Chen–Li–Ou integral form of the method of moving planes, we prove that positive solutions to this system are radially symmetric and radially decreasing with respect to the same point when $p, q \geq 2$. Additionally, we obtain a classification result for positive action minimizers in the case $p = q$ and $\tau = \eta$.

1 Introduction

This work is focused on qualitative properties and the classification of positive solutions of the following coupled nonlocal semilinear system:

$$\begin{cases} -\Delta u + \tau u = \frac{2p}{p+q}\left(I_\alpha * |v|^q\right)|u|^{p-2}u; \\ -\Delta v + \eta v = \frac{2q}{p+q}\left(I_\alpha * |u|^p\right)|v|^{q-2}v \end{cases} \quad \text{in} \quad \mathbb{R}^N, \tag{1}$$

where $N \in \mathbb{N}$; $0 < \alpha < N$; $\tau, \eta > 0$; $I_\alpha \colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ denotes the *Riesz potential* defined as

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha}|x|^{-(N-\alpha)};$$

$p, q > 1$ and the unknowns are $u, v \colon \mathbb{R}^N \to \mathbb{R}$.

We will suppose that the parameters p, q satisfy the following condition:

$$\max\left\{1, \frac{2\alpha}{N}\right\} < p, q < 2^* \quad \text{and} \quad \frac{2(N+\alpha)}{N} < p + q < 2^*_\alpha, \tag{H_1}$$

where

$$2^* := \begin{cases} \infty, & \text{if } N \in \{1, 2\}, \\ \frac{2N}{N-2}, & \text{if } N \geq 3, \end{cases} \quad \text{and} \quad 2^*_\alpha := \begin{cases} \infty, & \text{if } N \in \{1, 2\}, \\ \frac{2(N+\alpha)}{N-2}, & \text{if } N \geq 3. \end{cases}$$

More precisely, (H_1) ensures that we can look for weak solutions to (1) in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ because the relevant integrals do not diverge due to the Sobolev embeddings and the Hardy–Littlewood–Sobolev inequality.

2 Main Results

2.1 Symmetry of positive solutions

Let us state our main result obtained by means of the Chen–Li–Ou integral form of the method of moving planes (see [3]).

Theorem 2.1. *Suppose that $N \in \mathbb{N}$; $0 < \alpha < N$; $\tau, \eta > 0$;*

$$2 \leq p, q < 2^*, \quad p + q < 2^*_\alpha \tag{H_2}$$

and $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is a weak solution to (1) such that $u, v \geq 0$ a.e. Then there exists $x_0 \in \mathbb{R}^N$ such that $u(\cdot - x_0)$ and $v(\cdot - x_0)$ are radial and strictly radially decreasing.

A similar result for the case of the Choquard-type equation

$$-\Delta u + \tau u = (I_\alpha * |u|^p)\,|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N \tag{2}$$

had already been obtained in [4] under hypotheses of difficult verification. As such, Theorem 2.1 also helps to clarify the domain of validity of the symmetry of positive solutions for the equation (2).

2.2 Classification of minimizers of the associated action functional

Consider the *action functional* $S\colon H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$S(u, v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + \tau u^2 + \eta v^2 \mathrm{d}\mu - \frac{1}{p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)\,|v|^q \mathrm{d}\mu. \tag{3}$$

It is natural that weak solutions to (1) in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ can be characterized as critical points of S and we can analogously obtain a variational characterization for weak solutions to (2) in $H^1(\mathbb{R}^N)$. An important type of special solutions, called *action minimizers*, refers to those solutions that minimize the associated action functional under the constraint of the Nehari manifold.

Our second main result shows that if $\tau = \eta$ and $p = q$, then positive action minimizers of (1) are in direct correspondence with positive action minimizers of (2).

Theorem 2.2. *Suppose that $N \in \mathbb{N}$, $0 < \alpha < N$, $\tau > 0$ and $1 + \frac{\alpha}{N} < p < \frac{2^*_\alpha}{2}$. Then (u, v) minimizes the action associated with (1) if, and only if, $u = v$ minimizes the action associated with (2).*

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XIX ENAMA - Novembro 2026 *83–84*

CRITICAL SCHRÖDINGER-BOPP-PODOLSKY SYSTEMS: SOLUTIONS IN THE SEMICLASSICAL LIMIT

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Abstract

The present work is directly related to the research line developed within the postdoctoral project and constitutes a solid foundation for the objectives currently being pursued in collaboration with Prof. Olímpio Hiroshi Miyagaki, affiliated with the Federal University of São Carlos (UFSCar). This project is support by the São Paulo Research Foundation (FAPESP). This work was carried out during my doctoral studies. We consider the following critical Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + Q(x)\phi u = h(x,u) + K(x)|u|^4 u & \text{in } \mathbb{R}^3 \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)u^2 & \text{in } \mathbb{R}^3 \end{cases}$$

in the unknowns $u, \phi : \mathbb{R}^3 \to \mathbb{R}$ and where $\varepsilon, a > 0$ are parameters. The functions V, K, Q satisfy suitable assumptions as well as the nonlinearity h which is subcritical. For any fixed $a > 0$, we show existence of "small" solutions in the semiclassical limit, namely whenever $\varepsilon \to 0$. We give also estimates of the norm of this solutions in terms of ε. Moreover, we show also that fixed ε suitably small, when $a \to 0$ the solutions found strongly converge to solutions of the Schrödinger-Poisson system. This result is part of the results obtained in [1].

1 Introduction

In the literature, this system was first introduced in [2]. It describes the stationary solutions of a charged particle in the electromagnetic field proposed by Bopp and Podolsky in the 1930s, which is regarded as more effective than the classical Maxwell theory of electromagnetism for short-range interactions. The system studied is the following in $\mathbb{R}^3$:

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2, \end{cases}$$

in the unknown functions $u, \phi : \mathbb{R}^3 \to \mathbb{R}$, where $a > 0$ is the parameter of the Bopp–Podolsky theory, $\omega > 0$ is the frequency of the stationary wave $\psi(x,t) = u(x)e^{-i\omega t}$, $q \neq 0$ is the charge of the particle, and $p \in (2,6)$ is a suitable parameter related to the interaction among many particles.

Motivated by the above work, we study the following critical semiclassical Schrödinger-Bopp-Podolsky system in $\mathbb{R}^3$:

$$\begin{cases} -\Delta u + \lambda V(x)u + \lambda Q(x)\phi u = \lambda h(x,u) + \lambda K(x)|u|^4 u \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)u^2. \end{cases} \tag{1}$$

Setting $\lambda = 1/\epsilon^2$, whose study, whenever $\lambda \to +\infty$ (equivalently, $\varepsilon \to 0$). Here $u, \phi : \mathbb{R}^3 \to \mathbb{R}$ are the unknowns and a is a suitable parameter. We assume that the real functions pV, Q, h, K satisfy the following:

V0) $V \in C(\mathbb{R}^3)$, $V \geq 0$, there is $x_0 \in \mathbb{R}^3$ such that $V(x_0) = 0$ and there is $b > 0$ such that meas$\{x \in \mathbb{R}^3 : V(x) < b\} < +\infty$;

Q0) it is $Q \geq 0$ ($\not\equiv 0$) and one of the following holds:

q1) $Q \in L^s(\mathbb{R}^3)$ for some $s \in (3, +\infty)$,

q2) $Q \in L^s(\mathbb{R}^3)$ for some $s \in [3/2, 3]$ and there exists $\lim_{x\to x_0} Q(x) = \ell \in \mathbb{R}$ (where x_0 is the one in (V0)),

q3) $Q \in L^\infty(\mathbb{R}^3)$ and for every $t > 0, \text{meas}\{x \in \mathbb{R}^3 : Q(x) \geq t\} < +\infty$;

K0) $K \in L^\infty(\mathbb{R}^3)$ and $0 < K_{\min} := \inf K \leq \sup K =: K_{\max} < \infty$;

H0) h1) $h \in C(\mathbb{R}^3 \times \mathbb{R})$ and $h(x, u) = o(|u|)$ uniformly in x as $u \to 0$;

h2) there exist $C_0 > 0$ and $q \in (2, 6)$ such that for any $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$ it holds $|h(x, u)| \leq C_0(|u| + |u|^{q-1})$;

h3) there exist $\nu > 0, p > 4$ and $4 < \mu < 6$ such that for any $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$ it holds

$$\nu\mu|u|^p \leq \mu H(x, u) \leq h(x, u)u, \quad \text{where } H(x, u) = \int_0^u h(x, s)ds$$

2 Main Results

Our results deal with existence of ground state, multiplicity and limit behaviour of the solutions in the semiclassical limit, namely as $\lambda \to +\infty$.

Theorem 2.1. *Let (V0), (Q0), (K0) and (H0) be satisfied and let $a > 0$. Then for λ sufficiently large system (1) has a positive solution of least energy $u_{\lambda,a} \in E$.*

Under the odd condition on the nonlinearity a multiplicity result is obtained. It is clear that in this case if $u_{\lambda,a}$ is a solution also $-u_{\lambda,a}$ is, so the solutions will always appear in pairs.

Theorem 2.2. *Let (V0), (Q0), (K0) and (H0) be satisfied and assume also that h is odd with respect to u. For any $m \in \mathbb{N}$ and $\sigma > 0$, there is $\Lambda_\sigma^m > 0$ such that for all $\lambda \geq \Lambda_\sigma^m$ and $a > 0$ system (1) has at least m (pairs of) solutions $u_{\lambda,a}$.*

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XIX ENAMA - Novembro 2026 *85–86*

SINGULAR ELLIPTIC PROBLEMS INVOLVING THE 1-LAPLACIAN WITH UNBOUNDED SUPERLINEAR TERMS

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Abstract

We investigate a class of singular elliptic boundary value problems driven by the 1-Laplacian operator under Dirichlet boundary conditions. The nonlinear structure combines a singular term of the type $\lambda u^{-\gamma}$ with a superlinear reaction given by $u^{g(r)-1}$, where g is a continuous radial function that may exhibit unbounded growth as $r \to 1^-$. The degenerate and nonsmooth nature of the 1-Laplacian, together with the presence of singular and possibly supercritical terms, precludes the direct use of standard variational techniques. Moreover, the presence of radial variable exponents introduces additional analytical challenges related to loss of homogeneity and superlinear growth. Our main result establishes the existence of two distinct positive radially symmetric solutions in a suitable parameter regime. These findings contribute to the theory of elliptic equations involving the 1-Laplacian with singular nonlinearities and further demonstrate the effectiveness of variational approximation methods in nonsmooth and degenerate elliptic settings.

1 Introduction

Singular elliptic equations driven by the 1-Laplacian have attracted considerable attention due to the analytical difficulties arising from the degeneracy of the operator and the presence of singular nonlinearities. Among the recent contributions, [1] established the existence of two positive solutions, for sufficiently small λ, to the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \dfrac{\lambda}{u^{\gamma}} + u^{q-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < 1^* = \frac{N}{N-1}$ is a constant exponent.

Motivated by this result, we investigate the more general problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \dfrac{\lambda}{u^{\gamma}} + u^{g(|x|)-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is the unit ball of $\mathbb{R}^N$, $\gamma \in (0,1)$, $\lambda > 0$, and $g : [0,1) \to \mathbb{R}$ is a continuous radial function.

The variable exponent gives rise to a genuinely nonhomogeneous model, preventing the direct application of standard variational methods. Our approach combines approximation through suitable p-Laplacian problems, comparison arguments, and a refined Moser iteration yielding uniform L^∞ estimates, which are crucial for passing to the limit as $p \to 1$. This allows us to establish a multiplicity result extending [1]. In particular, our assumptions allow the exponent to be unbounded from above, namely,

$$\lim_{r \to 1^-} g(r) = +\infty,$$

thereby considerably enlarging the class of admissible nonlinearities while recovering the constant exponent case as a particular situation.

For general references on elliptic problems driven by the 1-Laplacian, we refer the reader to [2, 3].

2 Main Results

Throughout this work, we assume that

$$g(0) < 1^* = \frac{N}{N-1} \quad \text{and} \quad g_- := \inf_{r\in[0,1)} g(r) > 1.$$

Our main result is the following.

Theorem 2.1. *Assume that g satisfies the above assumptions and let $\gamma \in (0,1)$. Then there exists $\lambda_* > 0$ such that, for every $\lambda \in (0, \lambda_*)$, problem (1) admits two distinct positive radially symmetric solutions.*

The above theorem extends the multiplicity result obtained in [1] from constant exponents to unbounded radial variable exponents.

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XIX ENAMA - Novembro 2026 *87–88*

KIRCHHOFF P-LAPLACIAN EQUATION WITH A NONLOCAL SINGULAR EXPONENT

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Abstract

This article establishes the existence of weak solutions for a class of p-Kirchhoff type problems with a nonlocal singular nonlinearity. By using Galerkin's method for the problem with the "frozen" exponent, combined with Schauder's fixed-point theorem and suitable energy estimates, we state the expected existence theorem.

1 Introduction

The purpose of this paper is to investigate the existence of weak solutions to the following problem:

$$\begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u \;=\; \dfrac{f(x)}{u^{\gamma(b(u))}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < \infty$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, M, f, b and γ are given functions, whose properties will be specified later.

The study of Kirchhoff-type problems, has a powerful background in various applications in physics and biology. For this reason, much attention has recently been given to the investigation of elliptic equations related to Kirchhoff coefficients, see [1,3] and the references therein. On the other hand, equations that include singularities have garnered significant interest because of their connection to models of non-Newtonian fluids, boundary layer phenomena for viscous fluids and chemical heterogeneous catalysts, see, e.g., [2]. Our main interest in this work is to extend the previous results to the case where the exponent of the singular term may depend on the space variable and the unknown solution u, see [4].

2 Assumptions and Main Result

We consider the following basic assumptions on the data for the study of the problem (1)

(A_1) $M : \mathbb{R} \to \mathbb{R}$ is continuous and there are constants $0 < m_0 \le m_\infty < \infty$ with $m_0 \le M(t) \le m_\infty$ for all $t \ge 0$.

(A_2) $\gamma : \mathbb{R} \to (0, 1)$ is continuous. If $1 < p < 2$ we additionally assume $\gamma(t) \le p - 1$ for all $t \in \mathbb{R}$ (with no extra assumption, whenever $p \ge 2$, since then $\gamma(t) < 1 \le p - 1$).

(A_3) $b : W_0^{1,\alpha}(\Omega) \to \mathbb{R}$ is continuous (i.e. $v_k \to v$ strongly in $W_0^{1,\alpha}(\Omega)$ implies $b(v_k) \to b(v)$ in $\mathbb{R}$) and bounded, in the sense that it maps every bounded subset of $W_0^{1,\alpha}(\Omega)$ into a bounded subset of $\mathbb{R}$; $1 < \alpha \le p$,

(A_4) $f \in L^m(\Omega)$, $f > 0$ a.e., with $m > N/p$ if $p < N$, and any $m > 1$ if $p \ge N$.

The main result in this work can be stated as follows.

Theorem 2.1. *Suppose* (A_1) - (A_4) *hold. Then problem* (1) *has a weak solution* $u \in W_0^{1,p}(\Omega)$, $u > 0$ *a.e. in* Ω.

Proof First we freeze the exponent of the singular term and apply the galerkin method to solve the problem then apply Schauder's fixed point theorem. Finally, we used a comparison result for the boundary passage and test the positivity of the solution. ∎

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XIX ENAMA - Novembro 2026 *89–90*

STUDY OF SINGULAR ELLIPTIC PROBLEMS INVOLVING THE P-LAPLACIAN

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Abstract

In this work, we study singular quasilinear elliptic problems involving the p-Laplacian operator, as part of the development of a master's thesis. The study takes as its starting point the work of Canino, Sciunzi, and Trombetta, which investigates existence and uniqueness results for singular problems driven by the p-Laplacian and featuring a singular term. In this initial stage of the research, we aim to understand the main results presented in the reference article and the techniques employed in their proofs. To obtain the existence results, procedures such as truncation and regularization, Schauder's fixed point theorem, a priori estimates, and approximation arguments are utilized. Uniqueness results are obtained through scaling arguments, asymptotic analysis, comparison principles, and regularization procedures. The study of the singular problem in bounded domains constitutes the first stage of the dissertation and provides the theoretical foundation for the subsequent stages of the research, which will involve unbounded domains.

1 Introduction

The p-Laplacian operator, defined by $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ for $1 < p < \infty$, plays an important role in the study of quasilinear elliptic partial differential equations. In this work, we consider the singular problem

$$\begin{cases} -\Delta_p u = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, $1 < p < \infty$, $\gamma > 0$, $f \geq 0$, and $f \not\equiv 0$. The nonlinearity $u^{-\gamma}$ introduces additional difficulties due to its behavior as u approaches zero. Because of this singularity, the solution must be considered in a suitable weak formulation and may not necessarily belong to the standard Sobolev space $W_0^{1,p}(\Omega)$.

This study is part of the master's thesis entitled *Study of Elliptic Problems Involving the p-Laplacian in Bounded and Unbounded Domains*. The fundamental paper by Canino, Sciunzi, and Trombetta [1] serves as the starting point for the first stage of our research, which is dedicated to the existence and uniqueness results for problem (1).

2 Main Results

Due to the singularity of the term $u^{-\gamma}$, a suitable notion of weak solution is required, where $u \in W^{1,p}_{\text{loc}}(\Omega)$, strict positivity is ensured in the interior of Ω, and the Dirichlet boundary condition is interpreted in a generalized sense.

The main existence results studied are presented as follows.

Theorem 2.1. *Suppose $\gamma \geq 1$ and $f \in L^1(\Omega)$. Then, there exists a weak solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to problem (1) such that*

$$u^{\frac{\gamma+p-1}{p}} \in W_0^{1,p}(\Omega). \tag{1}$$

Furthermore, for $0 < \gamma < 1$ *and* $1 < p < N$*, if* $f \in L^m(\Omega)$ *with*

$$m = \frac{Np}{N(p-1) + p + \gamma(N-p)} \left(\frac{p^*}{1-\gamma} \right)', \tag{2}$$

where $(p^*)'$ *is the Hölder conjugate of the Sobolev critical exponent, then there exists a solution* $u \in W_0^{1,p}(\Omega)$*. Similar integrability conditions on* f *ensure existence for* $p = N$ *and* $p > N$*.*

The proof relies heavily on approximation and regularization techniques. The singular term is truncated, and Schauder's fixed point theorem is applied to the regularized problems. A priori estimates and convergence arguments (using tools from [3]) are then used to pass to the limit and obtain a solution to the original singular problem.

Regarding the uniqueness of solutions, the geometry of the domain plays a crucial role in the initial approach.

Theorem 2.2. *Let* $\gamma > 0$*,* $f \in L^1(\Omega)$*, and suppose that* Ω *is a star-shaped domain with respect to the origin. Then, the weak solution to problem* (1) *is unique.*

The proof of Theorem 2.2 utilizes a scaling argument combined with asymptotic analysis and a comparison principle. For general domains, uniqueness can still be recovered provided that higher integrability conditions are imposed on the source term f, as detailed in [1]. These general uniqueness proofs involve carefully constructed regularized test functions and comparison arguments similar to those found in semilinear cases [2].

3 Future Developments

The detailed study of the aforementioned existence and uniqueness results in bounded domains [1] establishes the theoretical foundation for the subsequent stages of this master's research.

In the upcoming phases of the project, we intend to study quasilinear singular problems defined in unbounded domains, specifically of the form

$$-\Delta_p u + V(x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1}$$

This stage will require investigating the existence and properties of solutions using variational methods and concentration-compactness techniques, in order to overcome the lack of compactness of the Sobolev embeddings in $\mathbb{R}^N$ and to handle the behavior of the potential $V(x)$.

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XIX ENAMA - Novembro 2026 *91–92*

SCHAUDER ESTIMATES FOR NON-CONVEX FULLY NONLINEAR ELLIPTIC PDES WITH DINI CONTINUOUS DATA

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Abstract

In this talk, we establish local Schauder estimates for flat viscosity solutions, that is, solutions with sufficiently small norms, to a class of fully nonlinear elliptic partial differential equations of the form

$$F(D^2u, x) + \langle \mathfrak{B}(x), Du\rangle = f(x) \quad \text{in} \quad \mathrm{B}_1 \subset \mathbb{R}^n, \tag{1}$$

where the operator F is differentiable, though not necessarily convex or concave. In addition, we impose suitable Dini-type continuity assumptions on the data. Our methodology is based on geometric tangential techniques, combined with compactness and perturbative arguments.

The results presented in this talk are part of my master's dissertation and are based on joint work with my advisor, Dr. João Vitor da Silva, and Dr. Junior da Silva Bessa. A detailed account of these results has been published in the *Bulletin of the Brazilian Mathematical Society* [1].
Keywords: Fully nonlinear elliptic PDEs, Schauder estimates, Dini continuity.

1 Introduction

Caffarelli (see e.g. [2, Theorem 8.1]) established Schauder estimates for the inhomogeneous problem $F(D^2u, x) = f(x)$ via a perturbation and compactness technique under suitable Hölder assumptions on data, and concavity or convexity assumption on the operator F.

For nearly twenty years, the question of whether *arbitrary* fully nonlinear elliptic equations $F(D^2\mathfrak{h}) = 0$ in B_1 (resp. $F(D^2\mathfrak{h}) = f(x)$) admit a general C^2 *a priori* regularity theory remained unresolved. This problem was finally settled by Nadirashvili and Vlăduţ's counterexamples to $C^{1,1}$ regularity in [5], which concluded this line of investigation. Their works, however, stimulated new research directions. In light of the inherent obstacles to developing a universal existence theory for classical solutions to fully nonlinear equations, a major focus of current research involves identifying supplementary structural or qualitative conditions on $F(\mathrm{X}, \cdot), F(\cdot, x), f$, and even on u that enable C^2 estimates.

2 Main Results

We make the following structural assumptions:

(A1) (**Uniform Ellipticity**) The operator $F : \mathrm{Sym}(n) \times \Omega \to \mathbb{R}$ is fully nonlinear and uniformly elliptic, with ellipticity constants $0 < \lambda \leq \Lambda$. Specifically, we require

$$\mathscr{P}^-_{\lambda,\Lambda}(\mathrm{N}) \leq F(\mathrm{M}+\mathrm{N}, x) - F(\mathrm{M}, x) \leq \mathscr{P}^+_{\lambda,\Lambda}(\mathrm{N}), \ \forall\, x \in \Omega, \ \forall\, \mathrm{M}, \mathrm{N} \in \mathrm{Sym}(n), \quad \text{whith}, \quad \mathrm{N} \geq 0. \tag{2}$$

(A2) (**Differentiability of the Nonlinearity**) We assume that $F \in C^1(\mathrm{Sym}(n))$, and there exists a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ such that

$$\|\mathrm{D_M}F(\mathrm{X},x) - \mathrm{D_M}F(\mathrm{Y},x)\| \le \omega(\|\mathrm{X}-\mathrm{Y}\|), \qquad \forall\, x \in \Omega \text{ and } \forall\, \mathrm{X},\mathrm{Y} \in \mathrm{Sym}(n).$$

(A3) (**Dini continuity in the L^n-sense**) There exists a modulus of continuity $\tau : [0,\infty) \to [0,\infty)$, and non-negative constants $\mathrm{C}_f, \mathrm{C}_{\theta_F}, \mathrm{C}_{\mathfrak{B}}$, such that

$$\left(\int_{\mathrm{B}_r(x_0)} |f(x)-f(x_0)|^{p_0}\,dx\right)^{\frac{1}{p_0}} \le \mathrm{C}_f\,\tau(r), \quad \left(\int_{\mathrm{B}_r(x_0)} |\theta_F(x,x_0)|^{p_0}\,dx\right)^{\frac{1}{p_0}} \le \mathrm{C}_{\theta_F}\,\tau(r), \quad \left(\int_{\mathrm{B}_r(x_0)} |\mathfrak{B}(x)-\mathfrak{B}(x_0)|^{p_0}\,dx\right)^{\frac{1}{p_0}} \le \mathrm{C}_{\mathfrak{B}}\,\tau(r)$$

where τ satisfies the Dini condition $\displaystyle\int_0^1 \frac{\tau(s)}{s}ds < \infty$ and $p_0 > n$.

(A4) (**Compatibility condition**) We further suppose the **nullity conditions**:

$$\liminf_{s\to 0^+} \sup_{0<r\le\frac{1}{2}} \frac{\tau(rs)}{\tau(r)} = 0 \quad \text{and} \quad \liminf_{s\to 0^+} \sup_{k\in\mathbb{N}} \frac{s^{\alpha_0}\tau(s^k)}{\tau(s^{k+1})} = 0 \text{ for some } \alpha_0 \in (0,1].$$

Within this framework, we are able to derive the following local Schauder estimates.

Theorem 2.1 (**Local $C^{2,\mathbf{Dini}}$ regularity**). *Let $u \in C^0(\mathrm{B}_1)$ be a viscosity solution to (1),*

where the assumptions (A1) – (A4) are in force. There exists $\delta_0 > 0$, depending only upon n, λ, Λ, ω, and $\tau(1)$, such that if

$$\|u\|_{L^\infty(\mathrm{B}_1)} \le \delta_0,$$

then $u \in C^{2,\psi}_{loc}(\mathrm{B}_1)$ and

$$\|u\|_{C^{2,\psi}(\mathrm{B}_{1/2})} \le \mathrm{C}_0 \cdot \delta_0,$$

where $\mathrm{C}_0 > 0$ depends only upon n, λ, Λ, α_0, ω, and τ and $\psi(t) = \tau(t) + \displaystyle\int_0^t \frac{\tau(s)}{s}ds$.

Our results can be viewed as an extension of the work by dos Prazeres and Teixeira [3, Theorem 2.2] and Kovats [4, Theorem 1], now within the framework of linear drift terms, non-convexity, and Dini continuity assumptions.

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XIX ENAMA - Novembro 2026 *93–94*

NONNEGATIVE SOLUTIONS TO A WEIGHTED INDEFINITE ELLIPTIC SCHRÖDINGER EQUATION

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Abstract

In this work, we consider the following weighted problem involving an indefinite nonlinearity:

$$(P_{\mu,\eta})\begin{cases} -\operatorname{div}(g(x)\nabla u) = \lambda_1\beta(x)u + \mu f(x,u) + W(x)h(u) + \eta u^{2_s^*-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(x) \geq 0 \text{ in } \Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\eta \in \{0,1\}$, $N \geq 3$, λ_1 denotes the first eigenvalue of the corresponding problem in Ω, $g^{-1} \in L^1_{loc}(\Omega)$ is a positive weight, and $\beta \in L^\infty(\Omega)$. The function W is indefinite and changes sign, and we impose suitable conditions on f and h. The constant 2_s^* is the critical exponent associated with the operator. Applying the mountain pass theorem, we prove the existence of a nonnegative solution for $(P_{\mu,\eta})$ and, moreover, obtain a pointwise lower bound for the first eigenvalue associated with the operator on an adequate domain.

1 Introduction

In recent years, significant attention has been given to the study of elliptic problems involving indefinite nonlinearities due to their relevance in both applied and theoretical mathematics. The presence of indefinite functions, which change sign within the domain, introduces unique challenges, particularly in ensuring the existence and multiplicity of solutions under suitable hypotheses. In this work, we focus on the weighted indefinite elliptic problem $(P_{\mu,\eta})$.

The Theorem 2.2 is derived from the ideas in [3, 2], in which the authors studied the critical problem

$$\begin{cases} -\Delta u = \lambda_1 u + \mu h(x,u) + W(x)f(u) + u^{\gamma-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(x) \geq 0 \text{ in } \Omega, \end{cases}$$

where $\gamma \geq 2^*$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, $N \geq 3$, $\mu > 0$ is a parameter, λ_1 is the first eigenvalue of the operator $(-\Delta, H_0^1(\Omega))$, and W is a nonlinear function that changes sign, under suitable conditions on f and h.

Let us consider

$$\mathcal{S} = \inf\left\{ \frac{\displaystyle\int_\Omega |\nabla u|^{2_s}\,dx}{\left(\displaystyle\int_\Omega |u|^{2_s^*}\,dx\right)^{\frac{2_s}{2_s^*}}} \;:\; u \in D^{1,2_s}(\mathbb{R}^N)\setminus\{0\} \right\}$$

and

$$\mathcal{S}_* = \inf \left\{ \frac{\left(\int_\Omega g(x)|\nabla u|^2 \, dx \right)^{\frac{1}{2}}}{\left(\int_\Omega |u|^{2_s^*} \, dx \right)^{\frac{1}{2_s^*}}} \ : \ u \in H_0^1(\Omega, g) \backslash \{0\} \right\}.$$

We also consider $\Omega_0 = \{x \in \Omega \mid W(x) = 0\}$, $|\Omega_0|$, the Lebesgue measure of the set Ω_0, and

$$\|g\|_{L_0^\infty} := \operatorname{ess\,sup} \{ \, |g(x)| \ : \ x \in \Omega_0 \} \, ,$$

2 Main Results

Under suitable conditions on g, β, f, h and W, presented in [1], the following results are proved:

Theorem 2.1. *There exists a positive number μ_1 such that the problem $(P_{\mu,0})$ has a nonnegative solution for all $\mu \in [0, \mu_1)$.*

Theorem 2.2. *If*

$$\|g\|_{L_0^\infty} |\Omega_0|^{\frac{2_s - 2}{2_s}} < \mathcal{S}_*^2 \mathcal{S}^{-\frac{2}{2_s}},$$

then there exists a positive number μ_1 such that the problem $(P_{\mu,1})$ has a nonnegative solution for all $\mu \in [0, \mu_1)$.

To conclude with our last main result, let us consider the following eigenvalue problem:

$$\begin{cases} -\operatorname{div}(g(x)\nabla u) = \lambda u & \text{in } \widehat{\Omega}, \\ u = 0 & \text{on } \partial\widehat{\Omega}, \end{cases} \tag{$\widehat{P}$}$$

where $\widehat{\Omega} \subset\subset (\Omega_+ \cup \Omega_0)$ is a domain with smooth boundary. Then we obtain the following result.

Theorem 2.3. *Denoting by $\lambda_1(\widehat{\Omega})$ the first eigenvalue of $(\widehat{P})$, we have*

$$\lambda_1(\widehat{\Omega}) > \lambda_1 \beta(x) \quad a.e. \ x \in \widehat{\Omega}.$$

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XIX ENAMA - Novembro 2026 *95–96*

MULTIPLICITY RESULTS FOR SUBCRITICAL CHOQUARD-TYPE SUPERLINEAR ELLIPTIC PROBLEMS WITH PARTIAL INTERFERENCE WITH THE SPECTRUM

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Abstract

In this work, we investigate the existence and multiplicity of nontrivial solutions for a class of non-homogeneous superlinear elliptic problems of Choquard type with Dirichlet boundary conditions. Specifically, our main focus lies on problem P_2, modeled by the equation:

$$-\Delta w - \lambda w - \left(\int_\Omega \frac{w_+^p(y)}{|x-y|^\mu}\,dy\right) w_+^{p-1} = h \quad \text{in } \Omega$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain, $0 < \mu < N$, the non-homogeneous term is of the form $h = te_1 + h_1$ (with $t > 0$ and $\int_\Omega h_1 e_1 = 0$), and e_1 is the first eigenfunction of the Laplacian. We assume that the parameter λ partially interferes with the spectrum of the Laplace operator, that is, $\lambda > \lambda_1$ and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$.

The main difficulty of our analysis lies in the interaction between the nonlocal Choquard term, the asymmetry of the nonlinearity (which acts only on the positive part w_+), the spectral interference, and the non-homogeneous term h. To overcome these obstacles, we study a transformed auxiliary problem P_1 using Rabinowitz's Generalized Mountain Pass Theorem.

The main results obtained guarantee that, in a specific subcritical regime restricted by the exponent:

$$1 < p < \frac{4\mu + 2 - N + \sqrt{(N-2)^2 + 8\mu(3N+2)}}{8\mu}$$

The proofs are strongly based on variational methods, the geometry of energy functionals, and fine interpolation estimates in Sobolev spaces adapted to the Riesz kernel.

Keywords: Choquard equation; Superlinear elliptic problems; Generalized Mountain Pass Theorem; Interference with the spectrum; Multiple solutions.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Dirichlet boundary, $N \geq 3$, $0 < \mu < N$, $\lambda > 0$, $1 < p < 2_\mu^*$, $h_1 \in C(\Omega)$ with $\int_\Omega h_1 e_1 = 0$, $\alpha > 0$ such that $-\alpha e_1 + \widetilde{h}_1 < 0$ where $\widetilde{h}_1 = (-\Delta - \lambda)^{-1} h_1$. We define the following problems:

$$\mathrm{P}_1 \begin{cases} -\Delta w - \lambda w - \left(\displaystyle\int_\Omega \frac{(w - \alpha e_1 + \tilde{h}_1)_+^p(y)}{|x-y|^\mu}\,dy\right)(w - \alpha e_1 + \tilde{h}_1)_+^{p-1} = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

$$\mathrm{P}_2 \begin{cases} -\Delta w - \lambda w - \left(\displaystyle\int_\Omega \frac{w_+^p(y)}{|x-y|^\mu}\,dy\right) w_+^{p-1} = h & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

where $h = te_1 + h_1$ with $t > 0$.

Use: Generalized Mountain Pass Theorem and a Lemma.

Theorem 1.1. *Let $E = W \oplus X$, where W is a finite-dimensional subspace, and let $I \in C^1(E, \mathbb{R})$ satisfying the Palais–Smale (PS) condition. Let $B_\sigma = \{x \in E \mid \|x\| < \sigma\}$. Suppose that I satisfies*

(a) there exist constants $\rho, \beta > 0$ such that $I|_{\partial B_\rho \cap X} \geqslant \beta$, and

(b) there exists a $v \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q := (\bar{B}_R \cap W) \oplus \{sv \mid 0 \leqslant s \leqslant R\}$, then $I|_{\partial Q} \leqslant 0$.

Then I possesses a critical value $c \geqslant \beta$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

where $\Gamma = \{h \in C(Q, E) : h(u) = u \text{ if } u \in \partial Q\}$.

Lemma 1.1. *Let $\lambda \in (\lambda_j, \lambda_{j+1})$, for some $j \in \mathbb{N}$. Then there exist $v \in E$ with $\|v\|_{L^2} = 1$ and an $R > 0$ satisfying the following:*

(I) $\displaystyle\int_\Omega \nabla v \cdot \nabla e_i \, dx = 0, \quad 1 \leqslant i \leqslant j,$

(II) $R^2(\lambda - \lambda_j) > \displaystyle\int_\Omega |\nabla v|^2 \, dx,$

(III) $(Rz + v)^+ \neq 0, \quad \forall z \in W = span\{e_1, \ldots, e_j\}$ *with* $\displaystyle\int_\Omega |z|^2 \leqslant 1.$

2 Main Results

Theorem 2.1. *If $\lambda > \lambda_1$ and $\lambda \neq \lambda_j \quad \forall j \in \mathbb{N}$ and $1 < p < \frac{4\mu+2-N+\sqrt{(N-2)^2+8\mu(3N+2)}}{8\mu}$, then Problem P_1 has at least one nontrivial solution.*

Theorem 2.2. *If $\lambda > \lambda_1$ and $\lambda \neq \lambda_j \quad \forall j \in \mathbb{N}$ and $1 < p < \frac{4\mu+2-N+\sqrt{(N-2)^2+8\mu(3N+2)}}{8\mu}$, then there exists a constant $T(h_1)$ such that for all $t > T(h_1)$ Problem P_2 has at least two nontrivial solutions.*

Lemma 2.1. *Let $N \geq 3$, $0 < \mu < N$. If $1 < p < \frac{4\mu+2-N+\sqrt{(N-2)^2+8\mu(3N+2)}}{8\mu}$, then:*

1. $\lambda_{N,\mu}(p) := \frac{2N-\mu-p(N-2)}{2N-p(N-2)}$ *is well defined and* $0 < \lambda_{N,\mu}(p) < 1$.
2. $|w|_{p_{t_{\mu,N}}} \leq |w|_p^{\lambda_{N,\mu}(p)} |w|_{2^*}^{1-\lambda_{N,\mu}(p)}$, *for all* $w \in H_0^1(\Omega)$
3. $0 < \frac{1}{\delta(p)} - 1$, *where* $\delta(p) := \left[1 - (\lambda_{N,\mu}(p)) \left(\frac{2p-1}{2p}\right)\right] (2p-1) > 0$

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OPTIMAL GRADIENT ESTIMATES FOR QUASILINEAR ELLIPTIC MODELS OF $P-$LAPLACIAN TYPE VIA BLOW-UP PROCEDURE

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Abstract

In this work, we establish optimal interior regularity and quantitative estimates for planar weak solutions to a class of quasilinear elliptic equations of p-Laplacian type with a field term and integrability assumptions on the source term. More precisely, we study solutions to

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x) + \operatorname{div} F \qquad \text{in } B_1 \subset \mathbb{R}^2, \tag{1}$$

in the degenerate regime $p > 2$, in connection with the planar regularity theory for p-harmonic and p-Poisson equations.

Our main result shows that the gradient of the solution is Hölder continuous with the optimal exponent, determined precisely by the interplay between the integrability of the scalar source term f and the Hölder regularity of the field term F. The approach is based on a blow-up procedure, combined with *a priori* estimates and Liouville-type rigidity arguments, which allow us to capture the natural scaling of the problem and identify the critical regularity exponent, in the spirit of optimal $C^{1,\alpha}$ estimates for quasilinear elliptic equations and recent developments on sharp planar regularity via *a priori* estimates.

In particular, in the absence of the field term, we recover exactly the optimal exponent predicted by the scaling of the equation and the regularity of the data. The result thus provides a quantitative formulation of optimal regularity for planar p-Laplacian-type models with integrability assumptions on the source term.

1 Main Results

We introduce the critical exponent

$$\alpha_* := \min\left\{\frac{1-\frac{2}{q}}{p-1}, \frac{\tau}{p-1}\right\}. \tag{2}$$

Our main result is the following optimal interior gradient estimate.

Theorem 1.1 (Optimal interior C^{1,α_*} estimate)**.** *Let $p > 2$, $q > 2$, and $0 < \tau \leq 1$. Assume that*

$$f \in L^q(B_1) \qquad \text{and} \qquad F \in C^{0,\tau}(B_1;\mathbb{R}^2).$$

Let $u \in W^{1,p}(B_1)$ be a weak solution to

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f + \operatorname{div} F \qquad \text{in } B_1. \tag{3}$$

Then

$$u \in C^{1,\alpha_*}_{\mathrm{loc}}(B_1).$$

Moreover, there exists a constant

$$C = C(p,q,\tau) > 0$$

such that

$$\|u\|_{C^{1,\alpha_*}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|^{\frac{1}{p-1}}_{L^q(B_1)} + [F]^{\frac{1}{p-1}}_{C^{0,\tau}(B_1)}\right). \tag{4}$$

When $F \equiv 0$, Theorem 1.1 yields the optimal exponent

$$\alpha_* = \frac{1 - \frac{2}{q}}{p-1}$$

for the planar p-Poisson equation.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
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SHARP REGULARITY FOR THE OBSTACLE PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS OF $P-$LAPLACIAN TYPE

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Abstract

In this talk, we establish regularity properties of weak solutions for the obstacle problem for a quasilinear elliptic equation of p-Laplacian type with varying coefficients and possibly unbounded source terms. Specifically, we obtain sharp gradient estimates, namely $C^{1,\alpha}_{loc}(B_1)$ estimates for the solution, with an explicit regularity exponent depending only on universal parameters. In addition, we devote particular attention to the study of the case $p = 2$ and $f \neq 0$. To the best of our knowledge, this aspect has not been addressed in the classical literature yet and plays an decisive role in our approach, thereby contributing to the analysis of the non-linear setting.

1 Introduction

In this talk, we will establish sharp regularity estimates for weak solutions of the obstacle problem for quasilinear elliptic equations of p-Laplacian type with varying coefficients:

$$\begin{cases} \operatorname{div}\, \mathfrak{a}(x,\nabla u) &= f(x) \quad \text{in} \quad \{u>\varphi\}\cap B_1 \\ \operatorname{div}\, \mathfrak{a}(x,\nabla u) &\leq f(x) \quad \text{in} \quad B_1 \\ u(x) &\geq \varphi(x) \quad \text{in} \quad B_1 \\ u(x) &= 0 \quad \text{on} \quad \partial B_1, \end{cases} \tag{1}$$

where $B_1 \subset \mathbb{R}^n$ denotes the unit n-dimensional ball with $n \geq 2$, $\varphi \in C^{1,\beta}(B_1) \cap \mathfrak{X}_{p,q}$, for $\beta \in (0,1]$ $f \in L^q(B_1)$, and $\mathfrak{X}_{p,q} := \left\{v \in W^{1,p}(B_1);\ \operatorname{div}\mathfrak{a}(x,\nabla v) \in L^q(B_1)\right\}$, for $q > n$ and $q \geq \frac{p}{p-1}$, where one must assume that $\varphi \leq 0$ on ∂B_1 in order for the problem to admit a nontrivial solution. Moreover, the vector field $\mathfrak{a} : B_1 \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular in the gradient variable, and it fulfills the following structural conditions (growth and ellipticity assumptions): for every $x, y \in B_1$ and $\xi, \eta \in \mathbb{R}^n$, the following holds

$$\begin{cases} |\mathfrak{a}(x,\xi)| + |\partial_\xi \mathfrak{a}(x,\xi)||\xi| &\leq \Lambda|\xi|^{p-1} \\ \lambda|\xi|^{p-2}|\eta|^2 &\leq \langle \partial_\xi \mathfrak{a}(x,\xi)\eta,\eta\rangle \\ \sup\limits_{\substack{x,y\in B_1 \\ x\neq y,\ |\xi|\neq 0}} \dfrac{|\mathfrak{a}(x,\xi)-\mathfrak{a}(y,\xi)|}{|x-y|^\sigma|\xi|^{p-1}} &\leq \mathfrak{L}_0 < \infty, \end{cases} \tag{2}$$

where $p \in (1,\infty)$, $\sigma \in (0,1]$, $0 < \lambda \leq \Lambda < \infty$, and $\mathfrak{L}_0 > 0$.

2 Main Results

To state our results, let $\alpha_{\mathrm{M}}^{(p)}$ denote the maximal Hölder exponent of the gradient for solutions to the homogeneous constant-coefficient equation (without the obstacle), and $\alpha_{\mathrm{M}}^{(p)^-}$ denotes any positive number strictly smaller than $\alpha_{\mathrm{M}}^{(p)}$. Define

$$\alpha_{\mathrm{E}} = \min\left\{\alpha_{\mathrm{M}}^{(p)^-}, \min\left\{\sigma, 1-\frac{n}{q}\right\}\cdot \min\left\{1, \frac{1}{p-1}\right\}\right\}.$$

Our first result establishes sharp regularity for the non-homogeneous obstacle problem in the case where $p = 2$.

Theorem 2.1. *Let $u \in H_0^1(B_1)$ be a weak solution to (1) satisfying the structural conditions for $p = 2$. Then, $u \in C_{loc}^{1,\iota_0}(B_1)$, where*

$$\iota_0 = \min\left\{\alpha_{\mathrm{M}}^{(2)^-}, \beta, \sigma, 1 - \frac{n}{q}\right\} \qquad \text{if } n < q \leq \infty. \tag{1}$$

Furthermore, the following regularity estimates hold

$$\|u\|_{C^{1,\iota_0}(B_{1/2})} \leq \mathrm{C}_0\left(n, \lambda, \Lambda, \alpha_{\mathrm{M}}^{(2)}, \beta, \sigma, q, \|f\|_{L^q(\Omega)}, \|\varphi\|_{C^{1,\beta}(\Omega)}\right).$$

In our second result, we obtain the existence, uniqueness, and sharp regularity for solutions of the obstacle problem governed by $p-$Laplacian-type operators.

Theorem 2.2. *Assume that the structural conditions are satisfied. Then, there exists a unique weak solution $u \in C_{loc}^{1,\alpha}(B_1) \cap W^{1,p}(B_1)$ to (1) with*

$$\alpha = \min\{\alpha_{\mathrm{E}}, \iota_0\} \tag{2}$$

Furthermore, the following regularity estimate holds at free boundary points,

$$\sup_{x \in B_r(y)} \frac{|u(x) - u(y) - (x-y)\cdot\nabla u(y)|}{r^{1+\min\left\{\iota_0, \min\left\{\sigma, 1-\frac{n}{q}\right\}\cdot\min\left\{1, \frac{1}{p-1}\right\}\right\}}} \leq \mathrm{C}_0\left(\|\varphi\|_{C^{1,\beta}(B_1)}^{p-1} + \|f\|_{L^q(B_1)}\right)^{\frac{1}{p-1}}, \tag{3}$$

for $y \in \partial\{u > \varphi\} \cap B_{1/2}$ and $r < \frac{1}{4}$, where $\mathrm{C}_0 = \mathrm{C}_0(n, p, q, \Lambda, \lambda, \|\mathfrak{a}\|_{C^{0,\sigma}(B_1)})$ is a universal constant.

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A QUASILINEAR ELLIPTIC EQUATION BEYOND THE TRUDINGER-MOSER GROWTH

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Abstract

We study a class of quasilinear elliptic equations in bounded domains of $\mathbb{R}^2$ involving the operator

$$-\Delta u - \kappa\Delta(u^2)u,$$

combined with a singular term and a nonlinear perturbation exhibiting exponential growth. More precisely, we consider the problem

$$-\Delta u - \kappa\Delta(u^2)u = \lambda u^{\alpha} + h(u) \quad \text{in}\Omega,$$

subject to homogeneous Dirichlet boundary conditions, where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\lambda > 0$, $\alpha \in (0,1)$, and $h : [0,\infty) \to \mathbb{R}$ is a continuous function with exponential growth. In contrast to the classical Trudinger-Moser framework, where the admissible growth is typically of order e^{s^2}, we allow nonlinearities exhibiting growth of the form $e^{\beta s^4}$. This extension is made possible by the quasilinear structure of the operator, which yields additional compactness properties and allows us to treat a broader class of nonlinear terms. Our approach relies on a suitable change of variables that transforms the original quasilinear problem into an auxiliary semilinear equation, making it possible to employ variational techniques. In addition, we investigate regularity properties of the corresponding weak solutions.

1 Introduction

We investigate the existence of positive solutions for the quasilinear elliptic problem

$$\begin{cases} -\Delta u - \kappa\Delta(u^2)u = \lambda u^{\alpha} + h(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $\lambda > 0$, $\alpha \in (0,1)$, and h is a continuous function satisfying

$$0 \le h(s)s \le C|s|^q e^{\beta s^4}, \quad \forall s \in \mathbb{R}, \tag{2}$$

for some constants $C > 0$, $q > 2$, and $\beta > 0$.

We emphasize that the nonlinearity may exhibit supercritical growth with respect to the classical Trudinger–Moser setting. While exponential nonlinearities of order e^{s^2} have been extensively studied, the case of growth of order $e^{\beta s^4}$ remains comparatively unexplored and requires different analytical tools.

A function $u \in H_0^1(\Omega) \cap C^{1,\sigma}(\overline{\Omega})$, $\sigma \in (0,1)$, is said to be a *weak solution* of (1) if

$$\int_\Omega (1+2u^2)\nabla u \cdot \nabla\phi\, dx + 2\int_\Omega |\nabla u|^2 u\phi\, dx = \int_\Omega \left[\lambda|u|^{\alpha-2}u + h(u)\right]\phi\, dx \tag{3}$$

for every $\phi \in H_0^1(\Omega)$.

The weak formulation above is well defined under assumption (2). Indeed, the regularity $u \in H_0^1(\Omega) \cap C^{1,\sigma}(\overline{\Omega})$ provides sufficient integrability to control the nonlinear terms through Sobolev embeddings and Hólder-type estimates.

2 Main Results

The main result of this work is the following.

Theorem 2.1. *Assume that h satisfies (2). Then there exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, problem (1) admits a positive weak solution $u_\lambda \in H_0^1(\Omega) \cap C^{1,\sigma}(\overline{\Omega})$, for some $\sigma \in (0,1)$. Moreover,*

$$\|u_\lambda\| \to 0 \quad \text{as } \lambda \to 0.$$

The proof is based on a transformation that converts the original quasilinear problem into a semilinear one. More precisely, we introduce the change of variable $v = f^{-1}(u)$, where f is defined by

$$\begin{cases} f'(t) = \dfrac{1}{\sqrt{1+2f(t)^2}}, & t \geq 0, \\ f(t) = -f(-t), & t < 0. \end{cases} \tag{1}$$

Setting $u = f(v)$, problem (1) becomes

$$\begin{cases} -\Delta v = \lambda\big(f(v)\big)^{\alpha} f'(v) + h\big(f(v)\big) f'(v), & \text{in } \Omega, \\ v > 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

with $v \in H_0^1(\Omega)$.

This reformulation allows us to exploit the variational structure of the auxiliary problem while preserving the essential features of the original quasilinear equation. The compactness properties induced by the transformation play a crucial role in handling nonlinearities whose growth exceeds the classical Trudinger-Moser threshold.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
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XIX ENAMA - Novembro 2026 *103–104*

CRITICAL EXPONENTIAL GROWTH IN QUASILINEAR ELLIPTIC PROBLEMS: A NEHARI MANIFOLD APPROACH

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Abstract

In this work, we investigate the quasilinear elliptic problem

$$-\Delta u - \frac{1}{2}\Delta\big(a(x)u^2\big)u + V(x)u = f(u), \quad x \in \mathbb{R}^2,$$

where V is a continuous potential and $f : \mathbb{R} \to \mathbb{R}$ is a continuous superlinear nonlinearity exhibiting critical exponential growth. By employing the Nehari manifold method as our main variational tool, we establish the existence of both nonnegative and nodal solutions. Our results extend and complement the classical work [2] of Jia-quan Liu, Ya-qi Wang, and Zhi-Qiang Wang by treating the critical exponential growth case.

1 Introduction

In this work, we establish the existence of nonnegative and nodal solutions for a class of quasilinear elliptic problems involving a continuous superlinear nonlinearity with critical exponential growth and a continuous potential V. More precisely, we consider

$$-\Delta u - \frac{1}{2}\Delta\left(a(x)u^2\right)u + V(x)u = f(u), \quad x \in \mathbb{R}^2 \tag{1}$$

Under the following nonlinearities hypotheses on the functions a, V and f are the following:

(a_0) $a \in C\left(\mathbb{R}^2, \mathbb{R}\right)$ and there exist $0 < a_0, a_1$ such that $a_0 < a(x) < a_1$, for all $x \in \mathbb{R}^2$;

(V_0) $V \in C\left(\mathbb{R}^2, \mathbb{R}\right)$;

(V_1) $0 < V_0 \leq \inf_{x\in\mathbb{R}^2} V(x)$;

(V_2) for all $M > 0$, there holds measure $\left(\left\{x \in \mathbb{R}^2 : V(x) \leq M\right\}\right) < \infty$;

(f_1) consider $f : \mathbb{R} \to \mathbb{R}$ is of class C^0 and there exists $\alpha_0 \geq 0$ such that the function $f(t)$ satisfies

$$\lim_{|t|\to\infty} \frac{f(t)}{\exp\left(\alpha|t|^2\right)} = 0 \text{ for } \alpha > \alpha_0 \text{ and } \lim_{|t|\to\infty} \frac{f(t)}{\exp\left(\alpha|t|^2\right)} = \infty \text{ for } \alpha < \alpha_0;$$

(f_2) the following limit holds: $\lim_{|t|\to 0} \frac{f(t)}{t^3} = 0$;

(f_3) the function $t \to \frac{f(t)}{|t|^3}$ is increasing in $\mathbb{R}\backslash\{0\}$;

(f_4) there are $r > 4$ and $\tau > \tau^*$ such that $\operatorname{sgn}(t)f(t) \geq \tau|t|^{r-1}$, for all $t \geq 0$, where

$$\tau > \tau^* := \max\left\{1, \left(\frac{4\alpha_0}{\pi} c_r \frac{r-2}{r-4}\right)^{(r-2)/2}\right\},$$

where c_r will be defined throughout the proof of results. The class of problems (1) has been extensively studied during the last years via variational methods. The main technique used is change of variable. For example, the authors in [3] used a dual approach to reduce the quasilinear equation to a semilinear one and used an Orlicz space framework. The same method was used in [1], where the normal Sobolev space was used as the working space. Since in our problem there is a function a, the change of variable used in [1] or [3] cannot be considered here. Then, it is necessary to work directly on the associate functional in spite of its lack of smoothness. To the best of our knowledge, the first existence result involving variational methods was obtained in [4]. In that paper the authors apply variational techniques to prove the existence of standing wave solutions for quasilinear Schrödinger equations containing strongly singular nonlinearities which include derivatives of the second order. Another pioneering work in sense is [2], in this paper the authors consider the existence of both one sign and nodal ground states of soliton type solutions for problem (1). Under certain conditions on the potential function V, the authors show that the problem has at least one weak positive solution as well as one sign-changing solution with exactly two nodal domains, respectively. The problem was treated, for both the positive solution and the sign-changing solution, by a more unified approach, that is the Nehari method.

2 Main Results

Theorem 2.1. *Assume that a satisfies* $(a_0), V$ *satisfies* $(V_0)-(V_2), f$ *satisfies* (f_1) *with* $\alpha_0 > 0$ *and* $(f_2)-(f_4)$ *and* $f(t) = 0$*, for* $t \leq 0$ *hold. Then, problem (1.1) has a nonnegative solution with minimal energy.*

Theorem 2.2. *Suppose that a satisfies* $(a_0), V$ *satisfies* $(V_0)-(V_2), f$ *satisfies* (f_1) *with* $\alpha_0 > 0$*,* $(f_2)-(f_4)$*. Then problem (1.1) possesses a least energy nodal solution, which has precisely two nodal domains.*

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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CRITICAL SYSTEM INVOLVING A BIHARMONIC AND P-LAPLACIAN OPERATOR

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Abstract

This work is concerned with the existence of a ground state solution for the following class of elliptic Kirchhoff-Boussinesq type problems given by

$$\begin{cases} \Delta(a(\varepsilon x)\Delta u) - \Delta_p u + u = Q_u(u,v) + \dfrac{1}{2_*}K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ \Delta(b(\varepsilon x)\Delta v) - \Delta_p v + v = Q_v(u,v) + \dfrac{1}{2_*}K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \\ u, v \in H^2(\mathbb{R}^N), \end{cases}$$

where $\varepsilon > 0$, $2 < p < 2^* = \frac{2N}{N-2}$, $2_* = \frac{2N}{N-4}$ and $N \geq 5$. We establish the existence of a ground state solution for the critical system by employing variational methods and we prove the multiplicity of solutions through the Ljusternik-Schnirelmann category.

1 Introduction

Fourth-order Schrödinger equations have attracted considerable attention due to their applications in several areas, including nonlinear optics, thin plate theory, fluid dynamics, free-boundary problems, and nonlinear elasticity. A general fourth-order nonlinear Schrödinger equation is given by

$$\begin{cases} i\partial_t u + \Delta(\alpha\Delta u) - \operatorname{div}(\theta\nabla u) + \overline{V}(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u(x,0) = \Psi(x) \quad \text{for each} \quad x \in \mathbb{R}^N, \ \Psi \in H^2(\mathbb{R}^N), \end{cases} \tag{1}$$

whose standing wave solutions $u(x,t) = e^{-ikt}v(x)$ lead to the stationary equation

$$\Delta(\alpha\Delta v) - \operatorname{div}(\theta\nabla v) + (\overline{V} + k)(x)v = f(x,v) \quad \text{in} \quad \mathbb{R}^N. \tag{2}$$

Replacing the Laplacian by the p-Laplacian in (2) yields stationary Kirchhoff-Boussinesq type equations, which arise in structural mechanics and describe the dynamics of elastic plates.

Problems of this type have been extensively investigated. Figueiredo and Carlos [1] established existence results for fourth-order quasilinear equations with non-periodic potentials by variational methods. Later, Figueiredo and Salirrosas [3] studied singularly perturbed biharmonic equations with subcritical and critical nonlinearities, proving existence and multiplicity of ground state solutions via variational methods and the Ljusternik-Schnirelmann category.

Motivated by these works, we study the following elliptic system

$$(S_\varepsilon) \qquad \begin{cases} \Delta(a(\varepsilon x)\Delta u) - \Delta_p u + u = Q_u(u,v) + \dfrac{1}{2_*}K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ \Delta(b(\varepsilon x)\Delta v) - \Delta_p v + v = Q_v(u,v) + \dfrac{1}{2_*}K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \\ u, v \in H^2(\mathbb{R}^N), \end{cases}$$

where $\varepsilon > 0$, $2 < p < 2^* = \frac{2N}{N-2}$, $2_* = \frac{2N}{N-4}$ and $N \geq 5$.

We assume that the potentials a and b are continuous, attain positive minimum values, and satisfy suitable asymptotic conditions. The nonlinearities Q and K are homogeneous functions satisfying subcritical and critical growth assumptions.

Defining

$$M = \{x \in \mathbb{R}^N : a(x) = a_0 \quad \text{and} \quad b(x) = b_0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\}, \quad \text{for } \delta > 0$$

our main result shows that the topology of M determines the number of solutions.

2 Main Results

The following theorem is the main result proved in [2].

Theorem 2.1. *Assume that conditions (a_1), (b_1) and $M \neq \emptyset$ hold. Suppose also that Q and K satisfy $(\mathcal{Q}_1)-(\mathcal{Q}_3)$. Then, for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ and $\sigma^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$ and for all $\sigma > \sigma^*$, the system (S_ε) admits at least $\text{cat}_{M_\delta}(M)$ nontrivial solutions.*

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *107–108*

A SUBDIFFERENTIAL APPROACH TO THE $P \to 1^+$ LIMIT OF FRACTIONAL SOBOLEV CONSTANTS

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Abstract

We study the asymptotic behavior, as $p \to 1^+$, of a nonlinear problem of the fractional p-Laplacian. More precisely, for a bounded open set $\Omega \subset \mathbb{R}^N$ and $0 < s < 1$, we consider

$$(-\Delta)_p^{s_p} u_p = \lambda_{p,q_p}^{s_p}(\Omega)|u_p|^{q_p-2}u_p \quad \text{in } \Omega, \qquad u_p = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $s_p = N + s - N/p$, $q_p = q(p) \to 1^+$, and $\lambda_{p,q_p}^{s_p}(\Omega)$ denotes the best constant for the embedding $W_0^{s_p,p}(\Omega) \hookrightarrow L^{q_p}(\Omega)$. We prove the convergence of the sharp constants and describe the limiting Euler–Lagrange equation in terms of the subdifferentials of the fractional $W^{s,1}$-seminorm and of the L^1 norm.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $0 < s < 1$. For $p > 1$ close to 1, define

$$s_p = N + s - \frac{N}{p}.$$

This choice is natural in the limit $p \to 1^+$, since

$$s_p \to s \quad \text{and} \quad s_p p = s + (N+s)(p-1) \to s$$

It also allows us to "rewrite" the Gagliardo seminorm as

$$[u]_{s_p,p}^p = \iint_{\mathbb{R}^{2N}} \left(\frac{|u(x) - u(y)|}{|x-y|^{N+s}} \right)^p dxdy.$$

Thus the problem can be understood as a passage from an L^p-type nonlocal energy to an L^1-type nonlocal energy, which allows one to identify the limit problem.

We define

$$\lambda_{p,q_p}^{s_p}(\Omega) := \inf_{u \in W_0^{s_p,p}(\Omega)\setminus\{0\}} \frac{[u]_{s_p,p}^p}{\|u\|_{L^{q_p}(\Omega)}^p}. \tag{1}$$

The limit, as $p \to 1^+$, goes to the fractional Cheeger constant

$$h_s(\Omega) := \inf_{E \subset \Omega,\ |E|>0} \frac{P_s(E)}{|E|},$$

where

$$P_s(E) := [1_E]_{s,1}.$$

This recovers the results for the corresponding local problem.

2 Main Results

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $0 < s < 1$, and let $q_p \to 1^+$. For every $p > 1$, let $u_p \geq 0$ be a minimizer of (1), normalized in $L^{q_p}(\Omega)$. Then, up to a subsequence,*

$$u_p \to u_1 \quad \text{strongly in } L^1(\Omega),$$

and

$$\lambda^{s_p}_{p,q_p}(\Omega) \to h_s(\Omega).$$

Moreover, there exist functions

$$z \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N), \qquad \xi_1 \in L^\infty(\Omega),$$

such that

$$z(x,y) = -z(y,x), \qquad |z(x,y)| \leq 1, \qquad z(x,y) \in \operatorname{sgn}(u_1(x) - u_1(y))$$

for a.e. $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, and

$$\xi_1(x) \in \operatorname{sgn}(u_1(x)) \quad \text{for a.e. } x \in \Omega.$$

The limiting Euler-Lagrange equation is

$$\iint_{\mathbb{R}^{2N}} \frac{z(x,y)(\varphi(x) - \varphi(y))}{|x-y|^{N+s}}\, dxdy = h_s(\Omega) \int_\Omega \xi_1 \varphi\, dx \tag{2}$$

for every admissible test function $\varphi \in W^{s,1}_0(\Omega)$.

The Euler-Lagrange equation of (2) is understood in the sense of a subdifferential inclusion. Here, $z(x,y)$ and $\xi_1(x)$ are subgradients of the Gagliardo seminorm $[\,\cdot\,]_{s,1}$ and of $\|\cdot\|_{q(p)}$, respectively.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *109–110*

MULTIPLICITY OF SOLUTIONS FOR A STATIONARY SCHRÖDINGER EQUATION VIA LUSTERNIK-SCHNIRELMANN THEORY

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Abstract

In this work, we combine concepts of differential geometry on Hilbert manifolds, critical point theory, and topological tools to study the category of Lusternik-Schnirelman used in counting solutions for nonlinear variational problems. In particular, we explore the study carried out by Cingolani and Lazzo [3] in 1997, which addresses the existence and multiplicity of standing waves $\psi(t) = e^{\frac{-i\lambda t}{\hbar}}u(x)$, with $\lambda \in \mathbb{R}$ and u a real-valued function, for the nonlinear Schrödinger equation in $\mathbb{R}^N$:

$$i\hbar\frac{\partial \psi}{\partial t} = -\hbar^2\Delta\psi + U(x)\psi + |\psi|^{p-2}\psi,$$

where $x \in \mathbb{R}^N$, i is the imaginary unit, $\hbar$ is Planck's constant, Δ is the Laplacian operator, e $p > 2$ if $N = 1, 2$ and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$. In this sense, we study positive solutions of the elliptic equation

$$-\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $V(x) = U(x) + \lambda$, $\lambda \in \mathbb{R}$, and $u(x) \to 0$ as $|x| \to \infty$. That is, we look for solutions to the elliptic problem given by

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u, \\ u > 0, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

1 Introduction

Initially, we will study the definition and important properties of the Lusternik-Schnirelmann category, which measures the topological richness of a set and allows us to estimate the number of critical points a functional can have. This mathematical concept will be used to prove the Lusternik-Schnirelmann Theorem for Hilbert spaces, which will later serve to establish a result guaranteeing the existence of multiple solutions for a stationary Schrödinger equation.

In this sense, we aim to find solutions to the problem

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u, \\ u > 0, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

that is, we are interested in obtaining the critical points of the functional associated with the problem given by

$$J_\varepsilon(u) = \int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + V(x)|u|^2)$$

restricted to the manifold $\Sigma = \left\{u \in \mathcal{H} : \int_{\mathbb{R}^N} |u|^p = 1\right\}$. Furthermore, the workspace is the Hilbert space $\mathcal{H} = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 < \infty\right\}$ embedded in the Sobolev space $H^1(\mathbb{R}^N)$, where the inner product incorporates the potential $V(x)$.

For this study, it is assumed that the potential $V(x)$ is continuous and satisfies the condition

$$\liminf_{|x|\to\infty} V(x) > V_0 > 0$$

such that $V_0 = \inf_{x\in\mathbb{R}^N} V(x)$. We use Lusternik-Schnirelmann Theory, which allows us to conclude that the topological richness of the set of global minima of the potential $V(x)$ determines the minimum number of distinct solutions for the elliptic problem in question. When referring to this topological richness of the set of minimum points, we mean the Lusternik-Schnirelmann category of the set of minimum points. This number provides a minimum number of weak solutions for the previously mentioned elliptic problem.

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XIX ENAMA - Novembro 2026 *111–112*

NONLINEAR FRACTIONAL MAGNETIC PROBLEM WITH AN ARBITRARY GROWTH AT INFINITY

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Abstract

We study the existence of radial solutions for a class of nonlinear fractional magnetic Schrödinger equations of the form

$$\varepsilon^{2s}(-\Delta)_A^s u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$ and $\varepsilon > 0$ is a small parameter. The magnetic potential A and the electric potential V are assumed to be continuous and radially symmetric, with V vanishing on an annular region.

The main novelty of this work lies in the treatment of nonlinearities with very general growth at infinity, including supercritical polynomial and exponential behaviors, which fall outside the scope of standard variational methods.

To overcome the lack of compactness inherent to this setting, we develop a variational approach combining truncation and penalization techniques in the fractional magnetic framework, together with radial compactness arguments.

As a result, we prove the existence of a nontrivial radial solution for sufficiently small ε, and we establish uniform L^∞ bounds and local Hölder regularity of the modulus of solutions. Our results extend and complement recent contributions on fractional magnetic equations, treating highly nonlinear regimes in a nonlocal magnetic context.

1 Introduction

In this work, we study the following class of nonlinear problems:

$$\epsilon^{2s}(-\Delta)_A^s u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $\epsilon > 0$, $N \geq 3$, and $s \in (0,1)$. The functions $V : \mathbb{R}^N \to \mathbb{R}$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ denote, respectively, the electric and magnetic potentials, while $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity.

The operator $(-\Delta)_A^s$ stands for the fractional magnetic Laplacian, defined by

$$(-\Delta)_A^s u(x) := c_{N,s} \lim_{r\to 0} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2s}}\, dy,$$

where the normalization constant $c_{N,s}$ is given by $c_{N,s} := \frac{4^s \Gamma\left(\frac{N-2s}{2}\right)}{\pi^{N/2}|\Gamma(-s)|}$.

Problems of the form (1) arise naturally in quantum mechanics when both long-range interactions and magnetic effects are taken into account. The presence of the fractional magnetic Laplacian reflects nonlocal interactions influenced by an external magnetic field, providing a natural extension of the classical magnetic SchrÃ¶dinger operator.

We assume that the potentials V, A and the function f satisfy the following conditions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is radial, that is, $V(x) = V(y)$ whenever $|x| = |y|$. Moreover, there exist positive constants $R_1 < r_1 < r_2 < R_2$ such that $V(x) = 0 \quad$ for all $x \in \Omega := \{z \in \mathbb{R}^N : r_1 < |z| < r_2\}$;

(V_2) $V(x) \geq V_0 > 0$ for all $x \in \Omega^c = B^c_{R_2} \cup B_{R_1}$, where $V_0 := \inf_{x\in\Omega^c} V(x)$.

(A_1) $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ and $A(x) = (0, 0, \ldots, 0)$ for all $x \in \Omega$;

(A_2) A is radial;

(f_1) There exists $\vartheta > 2$ such that $0 < \vartheta F(t) = \vartheta \displaystyle\int_0^t f(s)\, ds \leq t f(t)$, for all $t > 0$;

(f_2) f is nondecreasing on $(0, +\infty)$;

(f_3) $\lim\limits_{t\to 0^+} f(t) = 0$ and $f(t) = 0$ for all $t \leq 0$.

2 Main Results

We are now in a position to state the main result of this work.

Theorem 2.1. *Assume that* (f_1)-(f_3), (V_1)–(V_2) *and* (A_1)-(A_2) *hold. Then, there exists* $\epsilon_0 > 0$ *such that, for every* $\epsilon \in (0, \epsilon_0)$, *problem* (1) *admits a nontrivial complex-valued solution* u_ϵ *whose modulus satisfies* $|u_\epsilon| \in L^\infty(\mathbb{R}^N) \cap C^{1,\lambda}_{\text{loc}}(\mathbb{R}^N)$, *for some* $\lambda \in (0, 1)$.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *113–114*

A NONLINEAR PARABOLIC PROBLEM INVOLVING A KIRCHHOFF-TYPE $P(B(U))\&Q(B(U))$-OPERATOR

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Abstract

This paper is concerned with the existence of weak solutions for a nonlinear parabolic problem driven by a Kirchhoff-type $p(b(u))\&q(b(u))$-operator. Our result is obtained using the Rothe time-discretisation method combined with a singular perturbation technique and Schauder's fixed-point theorem.

1 Introduction

In this work we discuss the existence of weak solutions for the following parabolic problem:

$$\begin{cases} u_t - M(\mathcal{A}(u))\,\mathrm{div}\left(a(|\nabla u|^{p(b(u))})|\nabla u|^{p(b(u))-2}\nabla u\right) + g(u) = f & \text{in } Q = \Omega \times]0,T[, \\ u = 0 & \text{on } \Sigma = \partial\Omega \times]0,T[, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary $\partial\Omega$; M, A, a, b, f and g are given functions whose properties will be stated later.

The stationary quasilinear equation associated with (1) was studied in [3, 4] under suitable conditions on the function a and the exponent $p(x,u,b(u)) = p(x)$. The cases $p(x,u,b(u)) = p(u)$, $M = 1$, $a(t) = 1 + t^{\frac{q-p}{p}}$ and $p(x,u,b(u)) = p(x,|\nabla u|)$, $M = 1$, $a(t) = 1+t^{\frac{q-p}{p}}$ were studied by Bahrouni et al. [1] and Fadil et al. [2], respectively. However, parabolic problems involving the $p(b(u))$-Laplacian operator have received comparatively less attention in the literature. In this work, we extend and complement the results of C. Zhang and X. Zhang, [5] and Hurtado [3] to the case where the operator is of $p(b(u))\&q(b(u))$-Kirchhoff type, this is with the exponent p and q depending on a nonlocal quantity $b(u)$.

2 Assumptions and Main Result

Throughout the paper we assume that: $p : \mathbb{R} \to [1, +\infty)$ is Lipschitz continuous and bounded, with $1 < \alpha < p(s) \leq \beta < \infty$, $N > q(s) > p(s) \geq 2, \forall\, s \in \mathbb{R}$, for some constants α, β. We write $\gamma(\lambda) := (1 - H(k_3))p(\lambda) + H(k_3)q(\lambda)$, where $H(k) = 1$ if $k > 0$ and $H(k) = 0$ if $k = 0$.

Let $b : W_0^{1,\alpha}(\Omega) \to \mathbb{R}$ be a continuous and bounded function, and $\alpha > \frac{2N}{N+2}$ (then $W_0^{1,\alpha}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$).

Since $p(b(u))$ and $q(b(u))$ are *real numbers* for each fixed u, all Sobolev spaces in this work are the classical ones. The solution space is

$$X(Q) := \left\{u \in L^\infty(0,T;L^2(\Omega)) \cap L^\alpha(0,T;W^{1,\alpha}(\Omega)) : \iint_Q |\nabla u|^{p(b(u))}\,dx\,dt < \infty\right\},$$

with norm $\|u\|_{X(Q)} := \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|_{L^\alpha(Q)} + \left(\iint_Q |\nabla u|^{p(b(u))}\,dx\,dt\right)^{1/\alpha}$.

Now, we give the following hypotheses.
(M_0). $M : [0, +\infty[\to]m_0, m_1[$ is continuous and increasing, with $m_0, m_1 > 0$.
(A_0). The function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is of class C^1 and there exist positive constants $k_0, k_1 > 0$ and $k_2, k_3 \geq 0$ such that

$$k_0 + H(k_3)\, k_2\, t^{(q-p)/p} \leq a(t) \leq k_1 + k_3\, t^{(q-p)/p}, \qquad \forall\, t \geq 0,$$

where $p = p(b(u))$ and $q = q(b(u))$; $\mathcal{A}(u) = \frac{1}{p(b(u))} \int_\Omega A\big(|\nabla u|^{p(b(u))}\big)\, dx$, $A(\xi) = \int_0^\xi a(s)\, ds$. .
(A_1). The function $h_a : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $h_a(t) := a(t^p)t^{p-2}$ is increasing.
(A_2). $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, continuous, $g(0) = 0$, $|g(s)| \leq C|s|$; (A_3). $f \in L^{\alpha'}(Q)$. The main result in this work can be stated as follows.

Theorem 2.1. *Suppose* (M_0), (A_0)- (A_3) *hold. For* $u_0 \in L^2(\Omega)$, *problem (1.1) has a weak solution* $u \in X(Q) \cap C([0, T]; L^2(\Omega))$.

Proof We use a the Rothe time-discretisation method combined with a singular perturbation technique and Schauder's fixed-point theorem.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *115–116*

DYNAMICS OF A NONLINEAR SWELLING POROUS THERMOELASTIC SYSTEM WITH SECOND SOUND

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Abstract

In this work we study a nonlinear swelling porous-elastic system coupled with a second sound heat system. We show that the problem is well-posed in suitable phase space. Moreover, using the quasi-stability technique developed by Chueshov and Lasiecka (2008), we also prove that the dynamical system generated by the problem possesses a regular global attractor, and that such attractor has finite fractal dimension.

1 Introduction

In this work, following [1, 2, 3], we consider the following nonlinear swelling porous thermoelastic system with second sound

$$\begin{cases} \rho_1\partial_t^2 z - a_{11}\partial_x^2 z - a_{12}\partial_x^2 u + g(\partial_t z) + f_1(z,u) = b_1(x), & \text{in} \quad (0,1)\times(0,+\infty), \\ \rho_2\partial_t^2 u - a_{22}\partial_x^2 u - a_{12}\partial_x^2 z + \delta\partial_x\theta + f_2(z,u) = b_2(x), & \text{in} \quad (0,1)\times(0,+\infty), \\ c\partial_t\theta + \partial_x q + \delta\partial_x\partial_t u = 0, & \text{in} \quad (0,1)\times(0,+\infty), \\ \tau\partial_t q + q + \kappa\partial_x\theta = 0, & \text{in} \quad (0,1)\times(0,+\infty), \end{cases} \tag{1}$$

with the following initial condition and mixed boundary condition respectively

$$\begin{cases} z(x,0) = z_0(x),\ \partial_t z(x,0) = z_1(x),\ \theta(x,0) = \theta_0(x), & x \in (0,1), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),\ q(x,0) = q_0(x), & x \in (0,1), \end{cases} \tag{2}$$

$$\begin{cases} z(0,t) = u(0,t) = q(0,t) = 0, & t \geq 0 \\ \partial_x z(1,t) = \partial_x u(1,t) = \theta(1,t) = 0, & t \geq 0. \end{cases} \tag{3}$$

2 Assumptions

We consider the following assumptions:

(i) For external body force, we suppose:

$$b_i \in L^2(0,1), \quad i = 1,2. \tag{4}$$

(ii) The nonlinear damping term satisfies $g \in C^1(\mathbb{R})$, $g(0) = 0$ and there exists positive constantes M_1, M_2 such that for each $s \in \mathbb{R}$

$$M_1 \leq g'(s) \leq M_2. \tag{5}$$

(*iii*) There exists a non-negative potential function $F \in C^2(\mathbb{R}^2)$ satisfying $\nabla F = (f_1, f_2)$, and there exist positive constants $\beta, m_F > 0$, such that

$$F(z,u) \geq -\beta(|z|^2 + |u|^2) - m_F \quad \text{with} \quad 0 < \beta < \frac{\lambda_1 \lambda_{\min}}{2}, \tag{6}$$

for any $z, u \in \mathbb{R}$, where $\lambda_{\min}$ is the smallest eingenvalue of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$. Here, as usual, $\lambda_1 > 0$ is the Poincarés constant.

(*iv*) There exist positive constants $p \geq 1$ and C_f such that

$$|\nabla f_i(z,u)| \leq C_f(1 + |z|^{p-1} + |u|^{p-1}) \tag{7}$$

(*v*) There exist a positive constant M_F such that

$$\nabla F(z,u) \cdot (z,u) \geq -\beta(|z|^2 + |u|^2) - M_F. \tag{8}$$

3 Main Results

Based on the quasi-stability theory developed by Chueshov-Lasiecka [4], the main result for the existence of global attractors is given by the following theorem:

Theorem 3.1. *Suppose that assumptions* (4)-(8) *hold and* $a_{11}a_{22} - a_{12}^2 > 0$. *Then, for the dynamical system* $(\mathcal{H}, S(\cdot))$ *generated by system* (1)-(3), *we have:*

(a) $S(t)$ *possesses a unique compact global attractor* $\mathcal{A} \subset \mathcal{H}$ *which is characterized by the unstable manifold* $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$ *of the set of stationary solutions of system* (1)-(3), *and having finite fractal dimension;*

(b) The attractor $\mathcal{A}$ *is bounded in* $\mathcal{H}^1 = H^2_*(0,1) \times H^2_*(0,1) \times H^1_*(0,1) \times H^1_*(0,1) \times H^1_0(0,1) \times H^1_*(0,1)$. *Moreover, every trajectory* $U(t) = (z(t), u(t), \partial_t z(t), \partial_t u(t), \theta(t), q(t))$ *in* $\mathcal{A}$ *satisfies*

$$eta, q??)\|^2_{H^1_\diamond \times H^1_*} + \|(\theta_t, q_t)\|^2_{(L^2)^2} \leq R_1^2, \tag{9}$$

for some constant $R_1 > 0$ *depending on* $\mathcal{A}$.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *117–118*

EXTENSIBLE PLATE EQUATION WITH THERMAL MEMORY AND EXPONENTIAL PERTURBATIONS

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Abstract

This work is concerned with long-time dynamics of solutions of extensible plate equations with thermal memory. The problem corresponds to a model of thermoelasticity based on a theory of non-Fourier heat flux. By considering the case where rotational inertia is present we show that the thermal dissipation is sufficient to stabilize the system under exponential nonlinear perturbations. We guarantee the existence of a finite-dimensional global attractor. In addition the existence of exponential attractors is also considered.

1 The problem

This work is concerned with long-time behavior of solutions for a class of semilinear problems related to thermoelastic extensible plates with Coleman-Gurtin heat law. More precisely, over a bounded domain $\Omega \subset \mathbb{R}^2$ with regular boundary Γ, we consider the following Initial Boundary Value Problem (IBVP):

$$u_{tt} + \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u - \Delta u_{tt} + f(u) + \nu\Delta\theta = h \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1}$$

$$\theta_t - \omega\Delta\theta - (1-\omega)\int_0^\infty k(s)\Delta\theta(t-s)ds - \nu\Delta u_t = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{2}$$

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+ \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,t)|_{t\le 0} = \theta_0(x,-t), \quad x \in \Omega, \tag{4}$$

where $\theta_0(x,t)$ is the prescribed history for $t \le 0$.

Under proper assumptions on the functions M and f, in [2] the authors prove that the IBVP generates a dynamical system $S(t)$ in a proper extended phase space $\mathcal{H}$. More specifically, it is assumed a polynomial-type growth for f in order to control the nonlinearity perturbation in the system. Here, in the present work, our goal is to generalize such a condition by allowing exponential growth for f. Then, by means of the Trudinge-Moser estimate, we still prove the problem (1)-(4) generates a dynamical system and analyze its long-time dynamics.

2 Assumptions and the Main Result

Let us assume that the following conditions:

1. $M \in C^1(\mathbb{R})$ and there exist constants $\alpha \in [0, \lambda_1)$ and $c_M \ge 0$ such that

$$M(s)s \ge -\alpha s - c_M \quad \text{and} \quad \hat{M}(s) \ge -\alpha s - c_M, \ \ \forall\, s \ge 0, \tag{5}$$

where $\hat{M}(z) = \int_0^z M(s)\,ds$ and $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$.

2. $h \in L^2(\Omega)$ and $f \in C^1(\mathbb{R})$ is such that for every $\gamma > 0$, there exist $C_\gamma > 0$ satisfying

$$|f'(z)| \leq C_\gamma e^{\gamma |z|^2}, \quad \forall\, z \in \mathbb{R}, \tag{6}$$

$$f(s)s \geq -\beta s^2 - c_f \quad \text{and} \quad \hat{f}(s) \geq -\frac{\beta}{2} s^2 - c_f, \ \ \forall\, s \in \mathbb{R}, \tag{7}$$

where $\hat{f}(z) = \int_0^z f(s)\, ds$, $c_f > 0$ and $\beta > 0$ is properly chosen.

Under the hypotheses (5)-(3), it is possible to prove that IBVP (1)-(4) is also well-posed on the same energy space $\mathcal{H}$, see e.g. [1], similar to what was done in [2]. Hence, the system also generates a dynamical system $(S(t), \mathcal{H})$ on a proper Hilbert space H.

Our main goal is to prove that $(\mathcal{H}, S(t))$ has the quasi-stability property, see e.g. the general theory raising from [3]. See also [4] for a more concise theory on quasi-stable dynamical systems. With such a property in hands, we are able to prove the following result.

Theorem 2.1. *Under the assumptions (5)-(3), we have:*

- *The associated dynamical system $(\mathcal{H}, S(t))$ has a compact global attractor $\mathcal{A}$.*
- *The global attractor $\mathcal{A}$ is characterized by the unstable manifold emanating from the set of stationary solutions.*
- *The global attractor $\mathcal{A}$ has finite fractal dimension.*
- *$(\mathcal{H}, S(t))$ has a T-discrete exponential attractor $\mathcal{A}_{\text{exp}}$ and $\mathcal{A} \subset \mathcal{A}_{\text{exp}}$.*

Proof. The complete proof is given in [1]. □

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *119–120*

MATHEMATICAL MODELING OF PARTICLE TRANSPORT IN POROUS MEDIA UNDER RAINFALL-INDUCED FLOW

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Abstract

Motivated by the history of intense rainfall events in the city of Juiz de Fora (Minas Gerais, Brazil), this study aims to investigate how the increase in groundwater flow velocity through the porous soil surrounding the aquifers supplying the city may influence the transport of fine particles detached from the porous matrix and subsequently contaminating the reservoirs. To address this problem, a mathematical model based on mass conservation was developed to describe particle transport in porous media. The proposed model incorporates the coupled effects of linear Henry adsorption and accounts for particle mobilization and immobilization mechanisms. The resulting transport equation was solved analytically using the method of characteristics for first-order hyperbolic conservation laws. The results were validated using numerical simulations with a finite-difference approximation, allowing a direct comparison between the two solutions. The proposed framework provides a mathematical basis for evaluating the influence of rainfall-induced flow conditions on particle transport and the potential contamination of water reservoirs by groundwater-induced particle transport.

1 Introduction

Our model describes the soil as a water-saturated porous medium (single-phase flow). Following the conservation law framework presented in [3], the particle mass balance is written as:

$$\phi\frac{\partial C}{\partial t}+\frac{\partial(UC)}{\partial x}+\frac{\partial\sigma_a}{\partial t}+\frac{\partial\sigma_s}{\partial t}=0, \tag{1}$$

where ϕ is porous medium porosity, C is suspended particle concentration, U is flow velocity, σ_a is concentration of particles adsorbed to the grain surfaces and σ_s is concentration of particles mobilized to the grain surfaces. There are two mechanisms of particles to be removed/added to the flow. The first mechanism is the linear Henry adsorption isotherm [4]:

$$\sigma_a=\lambda_a C, \tag{2}$$

where λ_a is adsorption filtration coefficient. The second mechanism is mobilization/immobilization purposed in [1]:

$$\frac{\partial\sigma_s}{\partial t}=-\lambda_s U\sigma_s-\lambda_{eq}\big(\sigma_s-\sigma_{eq}\big)\ , \tag{3}$$

where λ_s is particle mobilization coefficient, λ_{eq} is particle immobilization coefficient, and σ_{eq} is equilibrium concentration of mobilized particles compatible with flow velocity U, defined by:

$$\sigma_{eq}(U)=\begin{cases}\sigma_0\left(1-\dfrac{U^2}{U_m^2}\right), & U<U_m,\\ 0, & U\geq U_m,\end{cases} \tag{4}$$

where U_m is the minimum velocity capable of mobilizing all mobilized particles and σ_0 is the maximum concentration of mobilized particles in the absence of flow. Finally, the initial and boundary conditions are:

$$C(x,0)=0,\quad C(0,t)=C_0,\quad \sigma_s(x,0)=\sigma_s^0,\quad \sigma_a(x,0)=0, \tag{5}$$

where C_0 is initial suspended particle concentration and σ_s^0 is initial mobilized particle concentration.

2 Main Results

The analytical solution was obtained by decomposing the problem into two flow regimes defined by the relation between the U and U_m, which governs the particle mobilization equilibrium function $\sigma_{eq}(U)$. In each regime, the mobilization dynamics of the mobilized particle concentration σ_s were solved independently, providing its temporal evolution from the initial condition σ_s^0. The resulting retention contribution was then incorporated into the mass conservation equation for the suspended particle concentration C, yielding a forced advective transport problem.

The concentration field $C(x,t)$ was determined through the method of characteristics [2], transforming the governing partial differential equation into ordinary differential equations along the characteristic trajectories. This approach provides an explicit description of particle migration while accounting for the effects of retention σ_s and release mechanisms. The adsorbed particle concentration σ_a, described through the linear equilibrium relationship with C, was subsequently obtained from the analytical concentration field.

The resulting formulation describes the coupled evolution of suspended particles C, mobilized particles σ_s, and adsorbed particles σ_a. The solution distinguishes between unperturbed regions and regions affected by the inlet concentration C_0, reflecting the propagation particle velocity.

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UFJF - Universidade Federal de Juiz de Fora
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INVISCID DAMPING AND LINEAR CHAOS FOR THE RAYLEIGH EQUATION USING C_0-SEMIGROUPS

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Abstract

This work investigates the long-time dynamics of perturbations around shear flows for the 2D incompressible Euler equations via the C_0-semigroup generated by the linearized Rayleigh operator. We outline the formal proof that this semigroup exhibits weak-strong Li-Yorke chaos, marking a novel occurrence in partial differential equations. Through asymptotic phase-space analysis, we demonstrate that the infinite growth of the H^s-norm and the non-decay of the L^2-norm preclude strong Li-Yorke chaos, while successfully recovering the fractional temporal decay rates characteristic of linear inviscid damping.

1 Introduction

The two-dimensional incompressible Euler equations on the cylinder $D = \mathbb{T} \times \mathbb{R}$ admit shear flow solutions of the form $(U(y), 0)$. For the Couette flow, $U(y) = y$, linear perturbations $\mathbf{u}$ with vorticity ω exhibit inviscid damping, where the H^s norm of the velocity field decays temporally while the vorticity grows. To resolve Sommerfeld's paradox regarding the theoretical stability versus physical instability of such flows, we consider the linearization of perturbations around general shear flows, leading to the Rayleigh equation.

2 Model

The linearized perturbation equation around the shear flows $U(y)$ in Euler equation is governed by the Rayleigh equation:

$$\partial_t \omega + U(y)\partial_x \omega - U''(y)\partial_x \Delta^{-1}\omega = 0, \quad \mathbf{u} = \nabla^{\perp}\Delta^{-1}\omega. \tag{1}$$

Assuming a specific family of traveling wave graphs for the vorticity level sets, the problem reduces to a functional equation $F[\lambda, f](\mathbf{x}) = 0$. The Fréchet derivative around the trivial state $f = 0$ defines the linear operator $\mathcal{L} = A + B$ where:

$$Ah = c(y)\partial_x h, \tag{2}$$

$$Bh = \frac{1}{4\pi}\int_{\mathbb{T}\times I_\epsilon} \varpi'(\bar{y})K(x-\bar{x}, y-\bar{y})\partial_x h(\bar{x},\bar{y})d\bar{x}d\bar{y} = \Delta^{-1}(\varpi' \cdot \partial_x h), \tag{3}$$

with $c(y) = \lambda + y - \int_0^y \varpi(\bar{y})d\bar{y}$ acting as the Shear basee flow, and $K(x, y)$ representing the Green's function for the Laplacian in $\mathbb{T} \times \mathbb{R}$.

The operator A governs purely transport dynamics and generates the C_0-semigroup $S_A(t)h_0(x, y) = h_0(x + c(y)t, y)$. Since $\|Bh\|_{H^{s+1}} \leq C\|h\|_{H^s}$, B is a bounded operator. By the bounded perturbation theorem, $\mathcal{L} = A + B$ generates a strongly continuous semigroup $S(t)$ in $H^s(D_\epsilon)$ for $s > 1$. The mild solution satisfies Duhamel's principle:

$$h(t) = S_A(t)h_0 + \int_0^t S_A(t-\tau)Bh(\tau)d\tau. \tag{4}$$

Applying the Fourier transform in x, the decoupled evolution for each wave number $k \neq 0$ is:

$$h_k(y,t) = e^{ikc(y)t}\left(h_k(y,0) + \int_0^t e^{-ikc(y)\tau}\widehat{(Bh)}_k(y,\tau)d\tau\right). \tag{5}$$

3 Obtained Results

The asymptotic behavior of the system is derived through bounds on the integral formulation in the phase space.

Theorem[Weak-Strong Li-Yorke Chaos for $S_A(t)$ and $S(t)$] For $h \in H^s(D_\epsilon)$ $(s > 1)$ with zero mean, the transport semigroup $S_A(t)$ satisfies:

1. Weak approximation: $\lim_{t\to\infty}\langle S_A(t)h, \phi\rangle_{H^s} = 0, \quad \forall \phi \in (H^s)^*$.
2. Strong separation: $\lim_{t\to\infty} \|S_A(t)h\|_{H^s} = \infty$.

Consequently, $S_A(t)$ exhibits weak-strong Li-Yorke chaos. By isolating perturbations where $Bh = 0$, the full semigroup $S(t)$ inherits this chaotic behavior.

The proof relies on the Riemann-Lebesgue lemma showing that highly oscillatory modes generated by $e^{ikc(y)t}$ vanish in the weak topology, while the chain rule $\partial_y S_A(t)h = \partial_y h + tc'(y)\partial_x h$ enforces unbounded growth in the norm topology.

Theorem[Decay for Operator B] Let $h \in H^s(D_\epsilon)$ with $s \geq 1$. Through sequential integrations by parts utilizing the regularity of the Green's kernel K_k, the non-local term decays as:

$$|\widehat{(Bh)}_k(y,t)| \leq C_k \langle t\rangle^{-\min(s,2)} \|h_k(\cdot,0)\|_{H^s(I_\epsilon)}. \tag{6}$$

Corollary[Non-Decay of the L^2 Norm] Integrating the bounded decay of $\widehat{(Bh)}_k$, the Duhamel formulation directly yields:

$$\lim_{t\to\infty} \|S(t)h\|_{L^2} = C > 0. \tag{7}$$

The strictly positive lower bound guarantees the impossibility of trajectories converging to zero in the norm topology, thus rigorously precluding strong Li-Yorke chaos.

Theorem[Infinite Growth of the H^s Norm] For $s > 1$, differentiating the Fourier formulation with respect to y and applying the temporal bounds of B yields the explicit divergence:

$$\lim_{t\to\infty} \|\partial_y h_k(\cdot,t)\|_{L^2} \geq |k\|h_k(\cdot,0)\|_{H^1} C \lim_{t\to\infty} \left(t - t^{2-s} - \ln(t)\right) = \infty. \tag{8}$$

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *123–124*

DECAY RATES OF C_0-SEMIGROUP: MLOG THEOREMS (BATTY-DUYCKAERTS) AND OPTIMALITY THEOREMS (BORICHEV-TOMILOV)

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Abstract

This work presents the main results studied during an Undergraduate Research project concerning the asymptotic behavior of linear evolution equations. The project investigates the deep relationship between optimal decay rates of bounded C_0-semigroups and the spectral properties of their infinitesimal generators along the imaginary axis. More precisely, we discuss in detail two landmark results in the asymptotic theory of operator semigroups: the $M_{\log}$ Theorem by C. J. K. Batty and T. Duyckaerts (Journal of Evolution Equations, v. 8, n. 4, p. 765-780, 2008), which establishes general upper bounds for semigroup decay in Banach spaces using complex analysis and Laplace transform techniques, and the Optimality Theorem by A. Borichev and Y. Tomilov (Mathematische Annalen, v. 347, n. 2, p. 455-478, 2010), which characterizes sharp polynomial decay rates in Hilbert spaces via resolvent growth estimates.

1 Introduction

An important question in the theory of differential equations refers to the asymptotic behavior (in time) of their solutions; more specifically, if they reach an equilibrium and, if so, with which speed. For linear PDEs that can be reformulated as abstract linear evolution equations, semigroup theory has proven to be an indispensable tool over the last few decades. Let $\mathcal{L}(X)$ the space of bounded linear operators on a Banach (or Hilbert) space X. For a densely defined linear operator A on X, let $\rho(A) := \{\lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \in \mathcal{L}(X)\}$ be the resolvent set, and let $R(\lambda, A) := (\lambda - A)^{-1}$ be the resolvent operator. The spectrum of A is denoted by $\sigma(A) = \mathbb{C} \setminus \rho(A)$. Consider the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, & x \in \mathcal{D}(A), \end{cases} \tag{1}$$

where X is a Banach space, A is a closed linear operator with domain $\mathcal{D}(A) \subset X$, and $u : [0, \infty) \to X$. Problem (1) is well-posed if and only if A generates a C_0-semigroup $(T(t))_{t\geq 0}$, in which case classical solutions are represented by $u(t) = T(t)x$. It is well established that the decay of solutions to (1) is closely connected to the spectral properties of the generator A. For bounded semigroups (i.e., $\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(X)} < \infty$), Batty (see [1] for more details) established that if $\sigma(A) \cap i\mathbb{R} = \emptyset$, then $\lim_{t\to\infty} \|T(t)A^{-1}\|_{\mathcal{L}(X)} = 0$, confirming that classical solutions decay asymptotically. A fundamental question concerns the rate at which this decay occurs. Obtaining optimal estimates for $\|T(t)A^{-1}\|_{\mathcal{L}(X)}$ as $t \to \infty$ based on the growth of the resolvent $\|R(is, A)\|_{\mathcal{L}(X)}$ as $|s| \to \infty$. By combining complex analysis techniques with Tauberian theorems for Laplace transforms, Batty and Duyckaerts [2] proved the $M_{\log}$ Theorem, establishing general decay bounds in Banach spaces. Furthermore, they conjectured that in Hilbert spaces, the logarithmic loss factor could be eliminated. In their groundbreaking paper, Borichev and Tomilov [3] proved this conjecture under polynomial resolvent growth, establishing a sharp equivalence between resolvent growth along the imaginary axis and polynomial energy decay in Hilbert spaces.

2 Main Results

First, we state the general decay result in Banach spaces established by Batty and Duyckaerts.

Theorem 2.1 (Batty-Duyckaerts, 2008). *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X with generator A such that $i\mathbb{R} \subset \rho(A)$. For each $t \geq 0$, define*

$$M(t) = \sup_{|s|\leq t} \|R(is, A)\|_{\mathcal{L}(X)} \quad \text{and} \quad M_{\log}(t) = M(t)\left(\log(1 + M(t)) + \log(1 + t)\right). \tag{1}$$

Then, there exist constants $B, C > 0$ such that, for all $t \geq B$,

$$\|T(t)A^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{M_{\log}^{-1}\left(\frac{t}{C}\right)}. \tag{2}$$

In the Hilbert space setting, Borichev and Tomilov removed the logarithmic factor, providing a complete characterization for polynomial decay rates.

Theorem 2.2 (Borichev-Tomilov, 2010). *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then, for any given $\alpha > 0$, the following assertions are equivalent:*

(i) There exist $C_1 > 0$ and $s_0 > 0$ such that for all $s \in \mathbb{R}$ with $|s| \geq s_0$,

$$\|R(is, A)\|_{\mathcal{L}(H)} \leq C_1|s|^{\alpha}; \tag{3}$$

(ii) There exist $C_2 > 0$ and $t_0 > 0$ such that for all $t \geq t_0$,

$$\|T(t)A^{-1}\|_{\mathcal{L}(H)} \leq C_2 t^{-1/\alpha}. \tag{4}$$

The proof of Theorem 2.2 relies heavily on Plancherel's theorem, complex interpolation, and fine spectral estimates that are exclusive to Hilbert spaces, showing that the bound (4) is sharp.

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *125–126*

CRITICAL BLOW-UP FOR A SHIFTED TRICOMI EQUATION WITH AN OSCILLATORY SCALE-INVARIANT MASS

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Abstract

We study finite-time blow-up for a semilinear shifted Tricomi equation with nonincreasing propagation speed and an oscillatory scale-invariant mass. At the Strauss-type critical exponent determined by the effective propagation, we prove that every finite-propagation energy solution with nonnegative, nontrivial, compactly supported data blows up in finite time. For sufficiently small data of size ε, we also establish a critical exponential lifespan upper bound $T_\varepsilon \leq \exp(C\varepsilon^{-p(p-1)})$. The proof combines two positive adjoint functions with complementary roles: a separated mode supplies the initial nonlinear lower bound, while a self-similar hypergeometric profile detects the logarithmic boundary effect specific to the critical case.

Keywords: blow-up; critical exponent; lifespan; shifted Tricomi equation; scale-invariant mass.

1 Introduction

We consider the Cauchy problem

$$\begin{cases} u_{tt} - (1+t)^{2\ell}\Delta u + \mu^2(1+t)^{-2}u = |u|^p, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = \varepsilon f(x), \qquad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $-1 < \ell \leq 0$, $\mu^2 > 1/4$, and $p > 1$. The coefficient $(1+t)^\ell$ gives a nonincreasing propagation speed, whereas $\mu^2(1+t)^{-2}$ is scale invariant. The inequality $\mu^2 > 1/4$ places the corresponding Cauchy-Euler equation in the so-called oscillatory regime.

For the classical semilinear wave equation, the Strauss exponent separates the small-data global-existence and blow-up regimes; Takamura and Wakasa [6] prove the sharp lifespan upper bound at the critical exponent in high dimensions. For generalized Tricomi equations, the threshold is modified by the propagation geometry; see He, Witt and Yin [1]. Lai, Palmieri, and Takamura [2] recently established critical blow-up for an Euler-Poisson-Darboux-Tricomi equation with scale-invariant damping and mass, assuming that the damping dominates or balances the mass. In contrast, (1) has no damping and contains an oscillatory scale-invariant mass, for which the corresponding temporal characteristic roots are complex. Related effects of scale-invariant damping and mass are studied in [5].

The critical exponent in our problem is the unique root $p = p_c(n,\ell) > 1$ of

$$\big((1+\ell)n - 1\big)p^2 - \big((1+\ell)n + 1 - 2\ell\big)p - 2(1+\ell) = 0, \tag{2}$$

which exists precisely when $(1+\ell)n > 1$. Our companion paper [3] treats the subcritical range. Here, we settle the endpoint, determine its lifespan scale, and show how the oscillatory mass can be incorporated into a positive test-function method.

2 Main result and strategy of the proof

The main result of the work is the following.

Theorem 2.1. *At the critical exponent $p = p_c(n, \ell)$, assume that $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are nonnegative, not both zero, and supported in B_R for some $0 < R < (1+\ell)^{-1}$. Then, every energy solution of (1) satisfying the natural finite-propagation property blows up in finite time. Moreover, for sufficiently small $\varepsilon > 0$, its lifespan T_ε satisfies*

$$T_\varepsilon \leq \exp\left[c\,\varepsilon^{-p(p-1)}\right]. \tag{1}$$

The proof uses two positive adjoint solutions with complementary roles. A separated mode combines a positive spherical eigenfunction with a positive decreasing temporal profile. On the propagation region, temporal decay compensates for spatial growth, and the resulting positive Wronskian gives the nonlinear lower bound inherited from the subcritical analysis in [3]. The complete proofs of these results announced here are given in [4].

The second adjoint is self-similar and hypergeometric. The oscillatory mass produces complex conjugate parameters, whose conjugate structure makes the power series coefficients nonnegative and hence preserves positivity. At the critical scaling, the profile remains bounded near the self-similar boundary, but its time derivative has a borderline singularity. Since the support stays strictly inside the cone, integration of this singularity yields precisely a logarithmic factor.

The critical identity (2) balances this logarithmic estimate with the lower bound from the separated mode. Weighted moments and logarithmic time then reduce the PDE to the differential inequality

$$\mathcal{J}''(\tau) + 2\mathcal{J}'(\tau) \geq c\tau^{1-p}\mathcal{J}(\tau)^p, \quad \text{with} \quad \mathcal{J}(\tau) \geq c\varepsilon^p\tau \quad \text{and} \quad \mathcal{J}'(\tau) \geq c\varepsilon^p. \tag{2}$$

Scaling by the critical time size $\varepsilon^{-p(p-1)}$ and comparing with a singular subsolution give (1). Thus, the Strauss-type mechanism persists, but the adjoint functions must encode both the changing propagation geometry and the oscillatory mass.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *127–128*

DIMENSIONAL REDUCTION AND NUMERICAL APPROXIMATION OF AN INHOMOGENEOUS STOKES MODEL IN A RECTANGULAR CHANNEL

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Abstract

We consider a scalar transient approximation motivated by inhomogeneous Stokes flow in a rectangular channel with wall friction. The model combines longitudinal diffusion, linear damping and a source representing body forces and pressure effects. We describe its explicit finite difference approximation and illustrate its transient response. The distinction between this effective model and the exact depth average of an incompressible flow is stated explicitly.

1 Model and averaging

Flow in drainage channels motivates the study of variable density and viscosity under slip boundary conditions [5]. Our starting point is the stationary two-dimensional Stokes framework of [1]. In $\Omega = (0, L) \times (0, H)$, let $\boldsymbol{u} = (u_1, u_2)$, p, ρ and μ denote velocity, pressure, density and the viscosity coefficient. The reference equations are

$$-\operatorname{div}(\mu D(\boldsymbol{u})) + \nabla p = \rho \boldsymbol{f}, \qquad \boldsymbol{u} \cdot \nabla \rho = 0, \qquad \operatorname{div} \boldsymbol{u} = 0, \tag{1}$$

where $D(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$. With the stress convention $\mu D(\boldsymbol{u})$, impermeability and Navier friction on the horizontal walls read

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad [\mu D(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = 0, \tag{2}$$

where $\boldsymbol{n}$ is the outward normal and $\alpha \geq 0$; see [5, 3]. For fixed height H, the depth average satisfies

$$\overline{u}(x,t) = \frac{1}{H} \int_0^H u_1(x, y, t)\, dy, \qquad \partial_x(H\overline{u}) = -u_2(x, H, t) + u_2(x, 0, t) = 0. \tag{3}$$

Thus a longitudinally varying scalar field cannot be identified with the exact mean velocity in this rigid incompressible channel. We retain the transient diffusion–reaction equation as an effective approximation, denoting its unknown by U:

$$\partial_t U = \nu \partial_{xx} U - \gamma U + S(x,t), \qquad \nu > 0, \quad \gamma \geq 0. \tag{4}$$

Here ν is a prescribed effective diffusivity and γ is a damping rate representing wall friction. For constant density, the source is motivated by $S = (\rho H)^{-1} \int_0^H (\rho f_1 - p_x)\, dy$. Time dependence, diffusion and damping are modeling assumptions, not consequences of the stationary system alone. The effective model is a classical parabolic equation [2, 4].

2 Numerical approximation

On a uniform grid $x_i = i\Delta x$, $t^n = n\Delta t$, let U_i^n approximate $U(x_i, t^n)$. The explicit FTCS scheme [4] is

$$U_i^{n+1} = U_i^n + \Delta t \left[\nu \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} - \gamma U_i^n + S_i^n \right]. \tag{1}$$

For constant coefficients, a sufficient condition for nonnegative stencil weights and maximum-norm stability of the homogeneous update is

$$\Delta t \left(\frac{2\nu}{\Delta x^2} + \gamma \right) \leq 1. \tag{2}$$

When $\gamma = 0$, this reduces to the usual diffusion restriction [4]. Figure 1 shows the transient profiles: the interior velocity grows from zero while both endpoint values remain zero. These curves illustrate (4); they do not validate a reduction of the full incompressible problem. A quantitative comparison requires the simulation parameters, initial

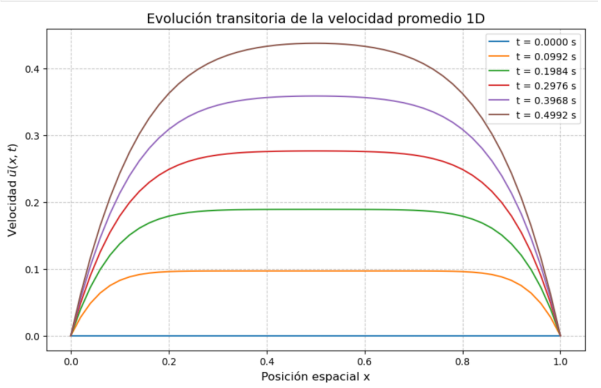


Figure 1: Transient profiles of the effective scalar model at the times shown.

and boundary data, and a compatible two-dimensional reference solution. Such a comparison is needed before making claims about reduction accuracy or computational savings.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *129–130*

ANALYTICAL MODELING OF FOAM-ASSISTED DISPLACEMENT WITH SURFACTANT ADSORPTION IN POROUS MEDIA

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Abstract

We develop an analytical framework for modeling foam-assisted displacement in porous media with surfactant adsorption. Formulated as a sequence of interacting Riemann problems for a system of conservation laws, our model accommodates sequential surfactant-gas injection and non-Newtonian foam rheology. We analyze the conditions under which the surfactant front reaches a finite asymptotic position.

1 Introduction

Foam injection is suitable for a wide range of applications, including Carbon Capture and Storage (CCS) and Enhanced Oil Recovery (EOR). By significantly increasing the apparent viscosity of the gas, foam promotes a more uniform displacement front, avoiding early breakthrough, gas fingering, and poor sweep efficiency. Among various injection strategies, Surfactant-Alternating-Gas (SAG) injection has gained popularity due to its operational advantages, such as improved injectivity and in situ foam generation. In this approach, water is followed by a surfactant slug and then gas.

This slug needs to be strictly designed to minimize the quantity of expensive chemicals used in the process and mitigate environmental impact. In this abstract, we summarize an analytical solution to a non-Newtonian foam displacement model [1] that considers both linear and concave surfactant adsorption models, utilizing an empirical foam model consistent with simulators like CMG-STARS [2].

2 Mathematical Model

We model the one-dimensional flow of two immiscible and incompressible phases (water and gas) through a fully saturated porous medium, neglecting capillary and gravitational effects. Foam and surfactant are treated as tracers, with foam impacting the gaseous phase and surfactant transported in the aqueous phase, where it may adsorb onto the rock surface.

Using dimensionless and normalized variables, the mass conservation equations for the water phase and the surfactant are reduced to a system of non-strictly hyperbolic partial differential equations:

$$\begin{aligned} \partial_t S + \partial_x f(S,C) &= 0, \\ \partial_t [C(S+\alpha) + A(C)] + \partial_x [f(S,C)C] &= 0, \end{aligned} \tag{1}$$

where S is the normalized water saturation, C is the normalized surfactant concentration, $f(S,C)$ is the aqueous fractional flow function, $A(C)$ is the normalized equilibrium adsorption function, and α is a scaling parameter related to connate water. The presence of foam reduces gas mobility, altering $f(S,C)$. This empirical approach also integrates non-Newtonian shear-thinning effects evaluated via the local capillary number.

3 Main Results and Discussion

The displacement problem is modeled as a sequence of interacting Riemann problems corresponding to the sequential injection of a surfactant slug followed by gas. When waves from the distinct Riemann problems intersect, the classical theory is adapted to trace the fronts, as illustrated in Figure 1.

One of the central findings is that the surfactant front does not propagate indefinitely; rather, it converges to a finite asymptotic spatial position over time. This limiting position is determined solely by the adsorption intensity and the injected slug parameters, entirely independent of the injection velocity or non-Newtonian foam rheology.

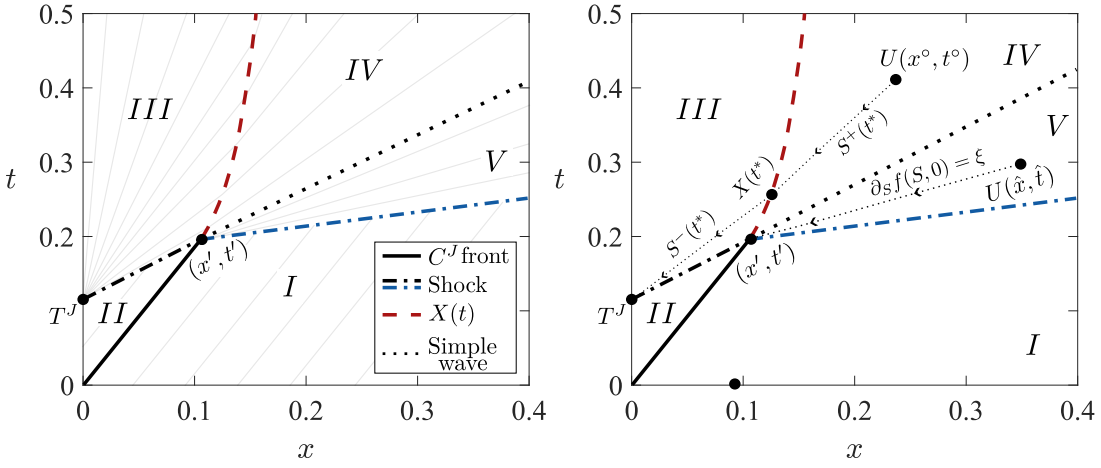


Figure 1: Schematic representation of the x-t plane, with characteristic curves shown as thin gray lines. The left panel displays the complete solution structure, where $C = C^J$ in regions II; III, and $C = 0$ in the remaining regions. The right panel illustrates two examples of solution construction: $U(x^\circ, t^\circ)$ and $U(\hat{x}, \hat{t})$. In Region V, we denote $\xi = (x - x')/(t - t')$.

Conversely, factors such as the injection velocity and the empirical shear-thinning parameter primarily affect transient dynamics. Higher velocities or more pronounced shear-thinning effects accelerate the interaction between the gas and surfactant fronts. This subsequently alters the timeline in which the pressure gradient evolves, though it does not change the ultimate penetration limit of the surfactant.

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XIX ENAMA - Novembro 2026 *131–132*

ROBUSTNESS OF NON-AUTONOMOUS ATTRACTORS FOR COUPLED PSEUDO-PARABOLIC SYSTEMS IN $\mathbb{R}^N$

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Abstract

We investigate the long-time dynamics of a class of non-autonomous coupled pseudo-parabolic systems posed in the whole space $\mathbb{R}^n$. These models describe diffusive processes with both parabolic and hyperbolic features, where pseudo-parabolic perturbations are governed by a small positive parameter. We prove the existence and uniqueness of weak solutions in the natural energy space and establish the existence of minimal non-autonomous pullback attractors in both the frameworks of fixed bounded sets and time-dependent tempered sets. Moreover, we establish relationships between the attractors generated in the different universes. Due to the lack of compact Sobolev embeddings in unbounded domains, the main compactness issues are addressed through a combination of uniform tail estimates and spectral decomposition techniques. Then, we study the continuity of the family of pullback attractors with respect to the perturbation parameter, proving continuity on a dense residual subset and upper semicontinuity as the perturbation parameter tends to zero.

1 Introduction

We study the asymptotic behavior of solutions to the following coupled pseudo-parabolic system posed on $\mathbb{R}^n$:

$$\begin{cases} u_t - \Delta u - \varepsilon \Delta u_t + \lambda u + f_1(u,v) = g_1(x,t), & t > \tau, \\ v_t - \Delta v - \varepsilon \Delta v_t + \lambda v + f_2(u,v) = g_2(x,t), & t > \tau, \\ u(x,\tau) = u_\tau(x), \quad v(x,\tau) = v_\tau(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\lambda > 0$, $\varepsilon \in [0,1]$ is a regularizing parameter, and $\tau \in \mathbb{R}$ denotes the initial time. The nonlinearities $f = (f_1, f_2)$ satisfy suitable growth and dissipativity conditions, and the external forces $g = (g_1, g_2)$ satisfy appropriate integrability and tail decay conditions. The parameter ε controls the strength of the pseudo-parabolic perturbation. For $\varepsilon > 0$, the terms $-\varepsilon\Delta u_t$ and $-\varepsilon\Delta v_t$ enhance the regularizing effect. When $\varepsilon = 0$, the system reduces to a classical reaction-diffusion model.

The main contributions are:

(i) Well-posedness in the phase space $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ via the Faedo-Galerkin method, using an approximation scheme on bounded domains to overcome the lack of compactness in $\mathbb{R}^n$.

(ii) Existence of minimal $\mathcal{B}$-pullback attractors $\mathcal{A}_B^\varepsilon$ and minimal $\mathcal{D}$-pullback attractors $\mathcal{A}_D^\varepsilon$ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, where $\mathcal{B}$ denotes the universe of fixed bounded sets and $\mathcal{D}$ consists of tempered time-dependent sets. Compactness is obtained through uniform tail estimates and spectral decomposition.

(iii) Robustness of the family of attractors with respect to ε: continuity on a dense residual subset of $[0,1]$, and upper semicontinuity at $\varepsilon = 0$. This requires establishing precompactness of $\bigcup_{\varepsilon\in[0,1]} \mathcal{A}_B^\varepsilon(t)$ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

2 Main Results

The nonlinear term $f = (f_1, f_2)$ satisfies:

Assumption 2.1. *f is conservative: there exists $F \in C^2(\mathbb{R}^2;\mathbb{R})$ such that $\nabla F = f$, and*

$$-\frac{\alpha}{2}(|u|^2+|v|^2) \le F(u,v) \le f(u,v)\cdot(u,v) + \frac{\alpha}{2}(|u|^2+|v|^2), \tag{2}$$

$$f_i(0,0)=0, \quad |\nabla f_i(u,v)| \le \beta(|u|^{p-1}+|v|^{p-1}), \quad i=1,2, \tag{3}$$

with $\beta>0$, $\alpha\in[0,\lambda)$, and $1\le p<\infty$ if $n=1,2$, and $1\le p<\frac{n+2}{n-2}$ if $n\ge 3$.

Assumption 2.2. *The external forces satisfy*

$$\int_{-\infty}^{t} e^{\frac{\sigma}{2(p+1)}r}(\|g_1(r)\|^2+\|g_2(r)\|^2)dr<\infty, \quad \forall t\in\mathbb{R}, \tag{4}$$

$$\lim_{k\to\infty}\int_{-\infty}^{t}\int_{|x|\ge k} e^{\sigma r}(|g_1(r)|^2+|g_2(r)|^2)dxdr=0, \tag{5}$$

where $0<\sigma<\min\{\mu,2/\varepsilon\}$, $\mu:=\lambda-\alpha>0$.

For upper semicontinuity, we require the stronger condition:

$$|\nabla f_i(u,v)|\le\beta_*, \quad \forall u,v\in\mathbb{R}, \quad i=1,2. \tag{6}$$

Theorem 2.1. *Under Assumptions (2.1) and 2.2, for any $(u_\tau,v_\tau)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$, problem (1) has a unique weak solution, depending continuously on the initial data.*

Theorem 2.2. *Under Assumptions 2.1 and 2.2, for each $\varepsilon\in[0,1]$, the process U_ε generated by (1) possesses:*

(i) a minimal $\mathcal{B}$-pullback attractor $\mathcal{A}_B^\varepsilon$ and a minimal $\mathcal{D}$-pullback attractor $\mathcal{A}_D^\varepsilon$ in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$, with $\mathcal{A}_D^\varepsilon\in\mathcal{D}$ and inclusions $\mathcal{A}_B^\varepsilon(t)\subset\mathcal{A}_D^\varepsilon(t)\subset D_0(t)$.

(ii) If additionally $\sup_{s\le 0}\int_{-\infty}^{s} e^{\sigma(r-t)}(\|g_1(r)\|^2+\|g_2(r)\|^2)dr<\infty$, then $\mathcal{A}_B^\varepsilon=\mathcal{A}_D^\varepsilon$.

Theorem 2.3. *Under Assumptions 2.1, 2.2, the stronger Lipschitz condition 6, and the temperedness condition above, assume also $\sup_{s\ge 0}\int_{-\infty}^{s} e^{\sigma(r-t)}(\|g_1(r)\|^2+\|g_2(r)\|^2)dr<\infty$. Then:*

(i) There exists a residual set $I\subset[0,1]$ such that for every $t\in\mathbb{R}$, the map $\varepsilon\mapsto\mathcal{A}_B^\varepsilon(t)$ is continuous at each $\varepsilon_0\in I$ in the Hausdorff symmetric distance.

(ii) The family $\mathcal{A}_B^\varepsilon$ is upper-semicontinuous at $\varepsilon=0$: $\lim_{\varepsilon\to 0}\operatorname{dist}_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}(\mathcal{A}_B^\varepsilon(t),\mathcal{A}_B^0(t))=0$.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *133–134*

ASYMPTOTIC BEHAVIOR OF A SEMILINEAR WAVE EQUATION WITH BOTH LOCALIZED MEMORY AND DELAY EFFECTS

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Abstract

In this work, we investigate a semilinear wave equation that incorporates localized memory and delay effects, allowing them to vanish in a set of positive measure. By imposing the δ-condition on the memory kernel and including a semilinear source term, we ensure that the energy of the system decays exponentially. While the viscoelastic dissipation is distributed around the boundary according to the Geometric Control Condition (GCC), the delay term is localized in such a way that it does not satisfy this condition. Our results demonstrate a flexible geometric interaction between memory and delay effects, ensuring stability even when instability occurs in uncontrolled regions.

1 Introduction

We study a semilinear wave equation with localized memory and delay effects given by the following initial and boundary value problem:

$$\begin{cases} \partial_t^2 u - \Delta u + \displaystyle\int_0^\infty k(s)\mathrm{div}[a(x)\nabla u(t-s)]ds + b(x)\partial_t u(x,t-\tau) + f(u) = 0, & \text{in} \quad \Omega\times(0,\infty), \\ u(x,t) = 0, & \text{on} \quad \Gamma\times\mathbb{R}, \\ u(x,-t) = u^0(x,-t), & \text{in} \quad \Omega\times[0,\infty), \\ \partial_t u(x,0) = \partial_t u^0(x,-t)|_{t=0}, & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded, and connected set with a smooth boundary Γ, k is the memory kernel, a and b are non-negative smooth functions responsible for the localized dissipation, and u^0 is the prescribed past history.

The combination of localized memory with a time delay term imposes severe analytical challenges. In general, the energy associated with this system is not strictly decreasing. To circumvent this problem, we rewrite the system in terms of Dafermos' relative history variable η (see [1]) and introduce a new variable z for the delay [3]. We construct an auxiliary problem containing modified damping terms, demonstrating that its corresponding energy functional is dissipative, and transfer the asymptotic behavior back to the original problem by using a bounded perturbation theorem (see [4]). These findings were recently published in the journal Asymptotic Analysis [2].

2 Main Results

The stability of the system requires us to deal with the spatial localization of the dissipation. The central difficulty arises when the dissipation associated with the delay, controlled by $b(x)$, acts only on regions that violate the Geometric Control Condition (GCC), while $a(x)$ fulfills it. We establish exponential decay by demonstrating a uniform observability estimate, employing a contradiction argument coupled with compactness techniques and microlocal analysis tools. In our analysis we assumed the following assumptions:

$(\mathbf{A_1})$ $a, b \in C^\infty(\Omega)\cap C^0(\overline{\Omega})$ are non-negative functions.

($\mathbf{A_2}$) $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-negative function such that $1 - \|a\|_\infty \int_0^\infty k(s)ds > 0$, and its derivative k' is an integrable, pointwise absolutely continuous, and non-increasing function.

($\mathbf{A_3}$) $f \in C^2(\mathbb{R})$ is a function such that $f(0) = 0$. Furthermore, its primitive $F(s)$ satisfies a suitable coercivity condition bounded by the first eigenvalue of the respective elliptic operator, and its derivatives satisfy critical growth conditions appropriate for the space dimension N.

($\mathbf{A_4}$) The memory kernel k satisfies the δ-condition, meaning there exist $\delta > 0$ and $C \geq 1$ such that $k(t + s) \leq Ce^{-\delta t}k(s)$ for every $t \geq 0$ and almost every $s > 0$. Additionally, k is not completely flat.

With these hypotheses established, the following theorems outline our key results:

Theorem 2.1 (Existence and Uniqueness)**.** *Suppose that the general structural assumptions* ($\mathbf{A_1}$)-($\mathbf{A_3}$) *hold. For any initial data $U_0 \in \mathcal{H}$, the problem possesses a unique global weak solution $(u, \partial_t u, \eta, z)$ such that*

$$u \in C([0,T]; H_0^1(\Omega)), \quad \partial_t u \in C([0,T]; L^2(\Omega)), \quad \eta \in C([0,T]; \mathcal{M}), \quad z \in C([0,T]; L^2(0,1; L^2(\Omega))).$$

Theorem 2.2 (Exponential Stability)**.** *Let $\xi > 1$, $R > 0$ be given constants. Under the assumptions* ($\mathbf{A_1}$) – ($\mathbf{A_4}$)*, there exists $T_0 > 0$ and constants $\widetilde{C}, \gamma > 0$ depending on $R > 0$ such that the energy of the original problem satisfies*

$$E(t) \leq \tilde{C}\mathcal{E}(0)e^{-\gamma t}, \quad \forall t \geq T_0, \tag{2}$$

provided that $\mathcal{E}(0) < \min\left\{R, \dfrac{\tilde{\gamma}}{\|b\|_\infty \xi e^\tau}\right\}$, where $\tilde{\gamma} = \tilde{\gamma}(U_0, \tau, \|b\|_\infty)$ is a suitable constant.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *135–136*

BOUNDARY AND INTERIOR CONTROL IN A DIFFUSIVE LOTKA-VOLTERRA MODEL

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Abstract

We investigate the controllability of a generalized diffusive Lotka-Volterra competition model for two species, incorporating boundary controls and an interior multiplicative control. Considering a smooth, bounded N-dimensional domain, we analyze ecologically pertinent scenarios characterized by constraints on both the controls and system states. Our results demonstrate how integrated control strategies can effectively overcome the limitations identified in previous studies. We prove two main results: (1) asymptotic controllability to single-species survival steady states under arbitrary system parameters, ensured by a combination of boundary and interior controls which act jointly to stabilize the system; and (2) finite-time controllability to a specific heterogeneous coexistence steady state via a two-phase strategy - first steering the system near the target with boundary control, then activating an interior multiplicative control in a localized region. The strong synergy between the two control mechanisms is crucial in both cases.

1 Introduction

This work is motivated by the need to develop rigorous mathematical results for the competitive Lotka-Volterra model (LVM), as such models are essential for understanding the ecological dynamics of interacting biological populations. These insights, in turn, are crucial for informed species conservation and ecosystem management. The version of the LVM analyzed captures competition between two species and is particularly well-suited for examining how limited resources drive interspecies interactions. We undertake a detailed analysis of these dynamics, supported by both mathematical arguments and numerical simulations.

We study a general diffusive LVM describing the dynamics of two competing species. The system involves two state variables, u and v, representing the respective population densities of the species. It is equipped with boundary conditions and features a multiplicative internal control. In our first result, concerning asymptotic controllability towards the survival of only one of the species, the internal control must act on the entire domain, and thus the problem is formulated as follows

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1u - c_1v) + hu, & (x,t) \in \Omega \times \mathbb{R}^+ \\ v_t = d_2\Delta v + v(a_2 - b_2u - c_2v), & (x,t) \in \Omega \times \mathbb{R}^+ \\ (u(x,0), v(x,0)) = (u_0, v_0), & x \in \Omega \\ u(x,t) = c_u(x,t), \quad v(x,t) = c_v(x,t), & (x,t) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1}$$

where a_i, b_i, c_i, d_i $(i = 1, 2)$ are positive parameters, and $\Omega \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ is a smooth domain with a regular boundary $\partial\Omega$.

Here, hu denotes the interior multiplicative control, while c_u and c_v are the boundary controls. The functions u_0 and v_0 represent the initial conditions of u and v, respectively. Given $T > 0$, we define the space-time cylinder $Q = \Omega \times (0, T)$ and its lateral boundary $\Sigma = \partial\Omega \times (0, T)$.

For the second result, concerning finite-time controllability towards a heterogeneous coexistence state, we replace the control term hu by the localized control $h1_\omega u$, where $\omega \subset \Omega$ is a nonempty open subset and 1_ω is its characteristic function.

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1u - c_1v) + h1_\omega u, & (x,t) \in \Omega \times \mathbb{R}^+ \\ v_t = d_2\Delta v + v(a_2 - b_2u - c_2v), & (x,t) \in \Omega \times \mathbb{R}^+ \\ (u(x,0), v(x,0)) = (u_0, v_0), & x \in \Omega \\ u(x,t) = c_u(x,t), \quad v(x,t) = c_v(x,t), & (x,t) \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{2}$$

Although the systems differ only in the support of the control, global in (1) and localized in (2), we present them separately for clarity and to facilitate the exposition of the results.

2 Main Results

For the uncontrolled system, the carrying capacities of u and v are a_1/b_1 and a_2/c_2, respectively. Therefore, it is natural to impose constraints on the solutions based on these values, i.e., $0 \le u \le a_1/b_1$, $0 \le v \le a_2/c_2$. This necessitates imposing the same constraints on the boundary controls $0 \le c_u \le a_1/b_1$, $0 \le c_v \le a_2/c_2$.

Theorem 2.1. *There are boundary controls $c_u, c_v \in L^\infty(\partial\Omega)$ and an interior control $h \in L^\infty(\Omega)$ such that, for every $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$, the solution (u, v) of (1) satisfies*

$$\lim_{t\to\infty} (u(\cdot,t), v(\cdot,t)) = (0, a_2/c_2)$$

uniformly in Ω.

Assume $d_1 = d_2 = d$, $a_1 = a_2 = a$, $b_1 > b_2$, and $c_1 < c_2$. Let θ be the positive solution of

$$\Big\{ d\Delta\theta + \theta(a - \theta) = 0, \ 0 < \theta < 1 \quad \text{in } \Omega, \ \theta \quad = 0 \quad \text{on } \partial\Omega,$$

and let $\lambda_1 > 0$ denote the first eigenvalue of the Dirichlet problem.

We are able to state our second result.

Theorem 2.2. *Let $\omega \subset \Omega$ be a nonempty open set. If $d < a/\lambda_1$, then for every $(u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$ there exist boundary controls $c_u, c_v \in L^\infty(\partial\Omega)$ and an interior control $h \in L^\infty(\omega \times (0,T))$ such that the solution (u, v) of (2) satisfies*

$$(u(\cdot,T), v(\cdot,T)) = (u^{**}, v^{**}) = \left(\left(\frac{c_2 - c_1}{b_1c_2 - c_1b_2} \right) \theta, \left(\frac{b_1 - b_2}{b_1c_2 - c_1b_2} \right) \theta \right)$$

for some $T > 0$.

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UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *137–138*

WEAKLY NONLINEAR MODELS FOR HYDROELASTIC WATER WAVES

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Abstract

In this talk, we present reduced interface models for hydroelastic water waves coupled to a nonlinear viscoelastic plate [3]. In a weakly nonlinear small-steepness regime we obtain bidirectional nonlocal evolution equations capturing the interface dynamics up to quadratic order. We also present a unidirectional model describing one-way propagation and we obtain local well-posedness for arbitrary data and global well-posedness for small data for this model.

1 Introduction

The interaction between incompressible fluids and deformable elastic structures stands as a challenging area in modern mathematical analysis of partial differential equations. Here we consider a three–dimensional incompressible, inviscid fluid evolving below a deformable elastic surface. This problem was originated in the works of Toland and Plotnikov [4]. In this context, one of the most prominent and physically relevant settings is hydroelasticity. Existing studies of hydroelastic waves are primarily devoted to traveling-wave solutions [5, 2, 6]. In two spatial dimensions the situation has also been extensively studied, most notably in the vortex-sheet formulation introduced by Ambrose and Liu [1]. In this talk, the authors obtained local well-posedness in Sobolev spaces and studied regularity mechanisms.

2 Main Results

We denote the one dimensional torus by $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Consider a three–dimensional incompressible, inviscid fluid, of constant density $\rho_f > 0$, evolving below a deformable elastic surface. The fluid occupies the time-dependent domain $\Omega(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -L\pi < x_1, x_2 < L\pi, \quad x_3 < \eta(t, x_1, x_2), \quad t \in [0, T]\}$, where $\eta(t, x_1, x_2)$ is the free surface and $L > 0$ denotes the typical horizontal wavelength, and the elastic free–boundary surface occupied by the plate is represented as the graph $\Gamma(t) = \{(x_1, x_2, \eta(x_1, x_2, t)) : (x_1, x_2) \in L\mathbb{S}^1 \times L\mathbb{S}^1, \quad t \in [0, T]\}$. Let ϕ be the potential velocity and denote by $\psi(x_1, x_2, t) = \phi(t, x_1, x_2, \eta(x_1, x_2, t))$ the surface potential. Then the fluid–structure interaction system admits the following potential formulation

$$\begin{aligned}
&\Delta\phi = 0 \text{ in } \Omega(t),\\
&\phi = \psi \text{ on } \Gamma(t),\\
&\partial_t\eta = \nabla\phi \cdot \sqrt{1 + |\nabla_x\eta|^2}\, n \text{ on } \Gamma(t),\\
&\partial_t\psi = -\tfrac{\rho_s h}{\rho_f}\,\partial_t^2\eta - \tfrac{1}{\rho_f}\,\mathcal{E}(\eta) + \tfrac{\gamma}{\rho_f}\,\partial_t\Delta_x\eta\\
&\qquad +\partial_{x_3}\phi\big(\nabla\phi \cdot \sqrt{1 + |\nabla_x\eta|^2}\, n\big) - \tfrac{1}{2}|\nabla\phi|^2 - g\eta \text{ on } \Gamma(t),
\end{aligned} \tag{1}$$

where, p, g denote, respectively,the pressure and the gravity. Here, as shown in [4], the fourth-order nonlinear geometric operator $\mathcal{E}(\eta)$ is given by $\mathcal{E}(\eta) = \frac{B}{2}\Delta_\Gamma H(\eta) + BH^3(\eta) - BH(\eta)K(\eta)$, where Δ_Γ is the Laplace–Beltrami operator and K is the Gauss curvature of the evolving surface

Providing the nondimensionalization of the system, a natural parameter $\varepsilon = H/L$ appears, where H denotes the typical vertical amplitude of the elastic sheet. Then, transforming the free-boundary problem into afixed reference domain $\Omega = \mathbb{T}^2 \times (-\infty, 0), \quad \Gamma = \mathbb{T}^2 \times \{0\}$, where $\mathbb{T}$ is identified with the interval $[-\pi, \pi]$, with periodic boundary conditions, using a time-dependent diffeomorphism $\Psi(\cdot, t) : \Omega \longrightarrow \Omega(t)$, defined by $\Psi(x_1, x_2, x_3, t) = \big(x_1,\ x_2,\ x_3 + \varepsilon\,\eta(x_1, x_2, t)\big)$, writing Φ, ψ and η as power series in ε and neglecting $\mathcal{O}(\varepsilon^2)$ terms, we obtain the following equation

$$\begin{aligned}&\big(I + \Upsilon\Lambda\big) f_{tt} + \delta\,\Lambda^3 f_t + \Big(\Lambda + \tfrac{\beta}{4}\Lambda^5\Big) f \\ &= \varepsilon\left[\tfrac{1}{2}\,\Lambda\Big((\partial_t f)^2 - \big|\mathcal{R}\partial_t f\big|^2\Big) - \partial_t\,\big([[\Lambda f]]\,\partial_t f\big) - \partial_t\Big(\nabla_x f \cdot \mathcal{R}\partial_t f\Big)\right],\end{aligned}$$

where $f = \eta^{(0)} + \varepsilon\eta^{(1)}$, $\Lambda := (-\Delta_x)^{1/2}$, $[[\Lambda f]]\partial_t f = \Lambda(f\partial_t f) - f\,\Lambda\partial_t f$ and where we use the Riesz transforms $\widehat{\mathcal{R}_j f}(k) = -i\frac{k_j}{|k|}\widehat{f}(k)$, $k \neq 0$, $j = 1, 2$, $\mathcal{R}f = (\mathcal{R}_1 f, \mathcal{R}_2 f)$. Then we are able to state the main theorem of this talk:

Theorem 2.1 (Local well-posedness for small H^3 data)**.** *Let $\Upsilon > 0$ and $\delta, \beta > 0$, and let $\varepsilon \in \mathbb{R}$ be fixed. Assume that the initial data satisfy $f(x, 0) = f_0 \in H_0^3(\mathbb{T}^2)$, $\partial_t f(x, 0) = f_1 \in H_0^1(\mathbb{T}^2)$, $\|f_0\|_{H^3} \leq \varepsilon_*$, for some $\varepsilon_* = \varepsilon_*(\Upsilon, \delta, \beta, |\varepsilon|) > 0$. Then there exists a time $T = T\big(\Upsilon, \delta, \beta, |\varepsilon|, \|f_1\|_{H^1}, \|f_0\|_{H^3}\big) > 0$ and a unique solution f of (2) on $[0, T]$ such that $f \in L^\infty(0, T; H_0^3(\mathbb{T}^2))$ and $f_t \in L^\infty(0, T; H_0^1(\mathbb{T}^2))$.*

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XIX ENAMA - Novembro 2026 *139–140*

GLOBAL STRONG SOLUTIONS AND DECAY ESTIMATES FOR A PENALIZED NONHOMOGENEOUS NEMATIC LIQUID CRYSTAL MODEL IN 3D THIN DOMAINS

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Abstract

In this work, we study a penalized Ericksen-Leslie system modeling variable density nematic liquid crystal flows in a three-dimensional thin domain $\Omega_\epsilon = \mathbb{R}^2 \times (0,\epsilon)$. The system consists of the density-dependent Navier-Stokes equations coupled with a parabolic equation for the director field, employing a Ginzburg-Landau penalization to approximate the constraint $|\boldsymbol{d}| = 1$. We establish the existence of global weak solutions using the Semi-Galerkin method. Furthermore, under a smallness condition on the initial data dependent on the thinness parameter ϵ, specifically $\epsilon^{1/2}\|\nabla \boldsymbol{u}_0\|_{L^2} \leq c_0$, we prove the existence and uniqueness of global strong solutions. Finally, we derive exponential decay estimates for the solution, revealing the stabilizing effect of the thin domain geometry.

1 Introduction

The hydrodynamic flow of nematic liquid crystals has been a subject of intense research due to its significant applications in physics and engineering. The continuum theory of liquid crystals was originally developed by Ericksen and Leslie during the 1960s [1]. We consider a simplified version of the Ericksen-Leslie system, proposed by Lin and Liu [2], which penalizes the constraint $|\boldsymbol{d}| = 1$ via a Ginzburg-Landau approximation, see also [5], [6].

Problems involving thin domains naturally arise in various physical contexts, such as tidal dynamics and meteorology. From a mathematical perspective, the study of fluids in thin domains $\Omega_\epsilon = \mathbb{R}^2 \times (0,\epsilon)$ has evolved from the classical Navier-Stokes framework [3] to more complex fluids, such as the variable density asymmetric fluids studied by Braz e Silva et al. [4]. The thinness of the spatial domain yields specific anisotropic PoincarÃ© and Sobolev inequalities, allowing us to establish global regularity results that remain an open problem in standard three-dimensional domains. We consider the following variable density PDE system in $Q_\epsilon = \Omega_\epsilon \times (0,T)$:

$$\begin{cases} \partial_t \rho + (\boldsymbol{u}\cdot\nabla)\rho = 0, \\ \rho(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) - \Delta \boldsymbol{u} + \nabla p = -\nabla \boldsymbol{d}^T \Delta \boldsymbol{d}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t \boldsymbol{d} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d} = \Delta \boldsymbol{d} - f(\boldsymbol{d}), \\ |\boldsymbol{d}| \leq 1, \end{cases} \tag{1}$$

with initial and boundary conditions $\rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x})$, $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x})$, $\boldsymbol{d}(\boldsymbol{x},0) = d_0(\boldsymbol{x})$ in Ω_ϵ, and $\boldsymbol{u} = 0$, $\boldsymbol{d} = \boldsymbol{h}$ on $\Gamma\times]0,\infty[$, where $0 < \alpha \leq \rho_0(\boldsymbol{x}) \leq \beta$, $f(\boldsymbol{d}) = \frac{1}{\sigma^2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}$ and $\Gamma := \{\boldsymbol{x} \in \bar{\Omega}_\epsilon : x_3 = 0 \text{ or } x_3 = \epsilon\}$.

2 Main Results

To treat the non-homogeneous boundary conditions for the director field, we employ a lifting function technique $\tilde{\boldsymbol{d}}$. Defining $\hat{\boldsymbol{d}} = \boldsymbol{d} - \tilde{\boldsymbol{d}}$, we recast the system with homogeneous boundary conditions. The main results regarding existence, uniqueness, and decay read as follows.

Theorem 2.1. *Assume* $\rho_0 \in L^\infty(\Omega_\epsilon)$ *with* $0 < \alpha \leq \rho_0 \leq \beta$, $(\boldsymbol{u}_0, \boldsymbol{d}_0) \in \boldsymbol{L}^2 \times \boldsymbol{H}^1(\Omega_\epsilon)$ *and compatible boundary conditions. If* $\boldsymbol{u}_0$ *satisfies the smallness condition* $\epsilon^{1/2}\|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega_\epsilon)} \leq c_0$, *with* $c_0 > 0$ *sufficiently small, then system* (1) *admits a global weak solution* $(\rho, \boldsymbol{u}, \boldsymbol{d}, p)$. *Furthermore, the weak solution satisfies* $0 < \alpha \leq \rho(\boldsymbol{x}, t) \leq \beta$ *and the following exponential decay estimate for all* $t \geq 0$*:*

$$\|\sqrt{\rho}\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2} + \|\nabla \hat{\boldsymbol{d}}(t)\|^2_{\boldsymbol{L}^2} \leq \left(C_1 + \|\sqrt{\rho_0}\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2}\right) e^{-\frac{t}{\epsilon^2}} + C_2\epsilon^2. \tag{1}$$

Theorem 2.2. *Assume the conditions of Theorem 2.1 and* $(\boldsymbol{u}_0, \boldsymbol{d}_0) \in \boldsymbol{V} \times \boldsymbol{H}^2(\Omega_\epsilon)$, *with* $\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_\epsilon) : \nabla \cdot \boldsymbol{v} = 0, \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\}$. *The system admits a unique global strong solution* $\boldsymbol{u} \in L^\infty(0, \infty; \boldsymbol{V}) \cap L^2(0, \infty; \boldsymbol{H}^2)$, $\hat{\boldsymbol{d}} \in L^\infty(0, \infty; \boldsymbol{H}^2) \cap L^2(0, \infty; \boldsymbol{H}^3)$, *and* $\nabla p \in L^2(0, \infty; L^2)$. *The unique strong solution satisfies the following exponential decay estimate for all* $t \geq 0$*:*

$$\|\nabla \boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2} + \|\Delta \hat{\boldsymbol{d}}(t)\|^2_{\boldsymbol{L}^2} \leq \left(C_3 + \|\nabla \boldsymbol{u}_0\|^2_{\boldsymbol{L}^2}\right) e^{-\frac{t}{\epsilon^2}} + C_4\epsilon^2. \tag{2}$$

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XIX ENAMA - Novembro 2026 *141–142*

BLOW-UP PREVENTION IN A PREDATOR-PREY MODEL WITH SINGULAR SENSITIVITY

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Abstract

In this work, we investigate a predator-prey model with singular sensitivity to chemotaxis in a regular bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} u_t - \Delta u + \chi \nabla \cdot \left(\dfrac{u}{p}\nabla p\right) = \alpha_1 uw - \alpha_2 u^2 - \alpha_3 u, & \text{in } (x,t) \in \Omega \times (0,T), \\ w_t - D_w \Delta w = \beta w \left(1 - w - \dfrac{u}{1+hu}\right), & \text{in } (x,t) \in \Omega \times (0,T), \\ -\Delta p = \delta_w w - \delta_p p, & \text{in } (x,t) \in \Omega \times (0,T), \end{cases} \tag{1}$$

assuming regular initial data and Neumann boundary condition. We establish the global well-posedness of the system for a range of model parameters and show that, regardless of parameter choices, the solutions remain uniformly bounded for all time. Furthermore, we conduct numerical experiments to explore the system's behavior and gain deeper insights into key aspects of the model.

1 Main Results

We start by stating the following local existence result, which is accompanied by a dichotomous criterion for deciding whether a solution can be extended to a global solution or if it blows up in finite time:

Lemma 1.1 (Local Existence)**.** *Let $\alpha_1, \alpha_3, \chi \in \mathbb{R}$, $\beta, \alpha_2, h \in \mathbb{R}_+$ and suppose $u_0, w_0 \in C^0(\overline{\Omega})$, are nonnegative and such that and $w_0 \not\equiv 0$. Then, there exist $T_{\max} > 0$ and a uniquely determined triple (u, w, p) of functions with $u, w \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times [0, T_{\max}))$ and $p \in C^{2.0}(\overline{\Omega} \times [0, T_{\max}))$ such that w and p are strictly positive in $\overline{\Omega} \times (0, T_{\max})$ and such that (u, w, p) solves the system (1) in the classical sense in $\Omega \times (0, T_{\max})$. If $T_{\max} < +\infty$ then*

$$\limsup_{t \nearrow T_{\max}} \|u(t)\|_\infty = +\infty. \tag{1}$$

Moreover, there exists a constant $\mathcal{M} > 0$, depending only on $\|u_0\|_1$, $\|w_0\|_1$, $|\Omega|$, and the coefficients α_1, α_2, α_3, β, such that

$$\|u(t)\|_1 + \|w(t)\|_1 \leq \mathcal{M}, \quad \forall t \in [0, T_{\max}). \tag{2}$$

For each $\tau \in (0, T_{\max})$ and $T \in (0, T_{\max}]$ there exists $\delta(\tau, T) > 0$ such that

$$p(x,t) \geq \delta(\tau, T), \quad \forall\ (x,t) \in \overline{\Omega} \times (0,T). \tag{3}$$

Finally, there exists $C_w > 0$ such that $0 \leq w(x,t) \leq C_w$ for all $x \in \Omega$ and $0 \leq t < T_{\max}$.

Proof. (Sketch) We start by invoking classical parabolic and elliptic results to ensure the existence of a solution $w[\phi]$ for the prey-equation and a solution $p = (I - \Delta)^{-1} w[\phi]$ for the pheromone-equation. We then define the set

$$S = \{\phi \in X,\ 0 \leq \phi(x,t) \leq R \text{ and } (I-\Delta)^{-1} w[\phi] \geq \varepsilon \text{ for all } t \in [0,T]\},$$

and apply heat semigroup estimates to prove that the map

$$\Phi(u) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\left(\chi\nabla\cdot\left(\frac{u(\cdot,s)}{p(\cdot,s)}\,\nabla p(\cdot,s)\right) + \alpha_1 u(\cdot,s)w[u](\cdot,s) - \alpha_2 u(\cdot,s)^2 - \alpha_3 u(\cdot,s)\right)\,ds,$$

is well-defined and also a contraction on S. Then, applying the Banach fixed point Theorem, we are able to establish the existence of a solution $u \in S$. Then using parabolic theory, parabolic strong maximal principle and elliptic strong maximal principle, we deduce that the triple (u, w, p) solves the system in the classical sense and that $u \geq 0, w > 0, p > 0$ hold in $\overline{\Omega} \times (0, T_{\max})$. ∎

Then, we prove L^γ estimates for u, and for every $t \in (0, T_{max})$ and $\gamma \geq 1$, to obtain a classical L^∞ estimation result:

Lemma 1.2. *Under the hypotheses of Lemma 1.1, the following holds: For every $\delta > 0$ and each $\tau \in (T, T_{\max})$, there exists a constant $C(\tau, \delta) > 0$ such that if, for some $T \in (\tau, T_{\max}]$, we have $p(x,t) \geq \delta$ for all $x \in \Omega$ and $t \in (\tau, T)$, then for any $\gamma \geq 1$ we have a constant C depending on δ, γ, $|\Omega|$, α_1, α_2, χ and $\|w\|_\infty$ such that*

$$\|u(x,t)\|_\gamma \leq C(\delta),$$

for all $t \in (\tau, T)$.

Lastly, applying the previous Lemma in the criterion for the finitude of T_{max} in Lemma 1.1, we are able to obtain a Global Existence result:

Theorem 1.1. *Let $\alpha_1, \chi \in \mathbb{R}$, $\alpha_2, \beta > 0$, $\alpha_3 \geq 0$, and suppose $u_0, w_0 \in C^0(\overline{\Omega})$ are nonnegative functions such that $w_0 \not\equiv 0$. Then, the system (1) admits a unique global classical solution (u, w, p) such that $u, w \in C^0(\overline{\Omega} \times [0, +\infty)) \cap C^{2,1}(\overline{\Omega} \times (0, +\infty))$, $p \in C^{2,0}(\overline{\Omega} \times (0, +\infty))$, and w, p are positive in $\overline{\Omega} \times (0, +\infty)$.*

Moreover, under these hypotheses, the prey population w cannot go extinct in finite time.

Proof. Suppose that $T_{\max} < \infty$. In Lemma 1.1 we obtain the lower bound

$$\|w\|_1 \geq e^C|\Omega|,$$

which allows us to determine $\delta > 0$ such that $p(x,t) > \delta$ for all $t \in (0, T_{\max})$. Thus, Lemma 1.2 applies, meaning that there exists a constant $C > 0$ such that $\|u(t)\|_\infty < C$ for all $t \in (0, T_{\max})$. This is a contradiction with the extensibility criterion (1). ∎

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ENAMA - Encontro Nacional de Análise Matemática e Aplicações
UFJF - Universidade Federal de Juiz de Fora
XIX ENAMA - Novembro 2026 *143–144*

BOUNDARY NULL CONTROLLABILITY OF DEGENERATE HEAT EQUATION AS THE LIMIT OF INTERNAL CONTROLLABILITY

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Abstract

In this work, we recover the boundary null controllability for the degenerate heat equation by analyzing the asymptotic behavior of an eligible family of state-control pairs $((u_\varepsilon, h_\varepsilon))_{\varepsilon\in(0,1)}$ solving corresponding singularly perturbed internal null controllability problems. As in other situations studied in the literature, our approach relies on Carleman estimates and meticulous weak convergence results. However, for the degenerate parabolic case, some specific trace operator inequalities must be obtained, in order to justify correctly the passage to the limit argument.

1 Introduction

Take $T > 0$, $\alpha \in (0, 2)$ and $\varepsilon \in (0, 1)$. Let us set

$$Q := (0, T) \times (0, 1), \quad \omega_\varepsilon := (1 - \varepsilon, 1) \text{ and } Q_\varepsilon := (0, T) \times \omega_\varepsilon.$$

Motivated by [3], [4] and [5], we investigate in [2] the existence of a family $(u_\varepsilon, h_\varepsilon)_{\varepsilon>0}$, solving the distributed null controllability problem

$$u_{\varepsilon t} - (x^\alpha u_{\varepsilon x})_x = h_\varepsilon \chi_{\omega_\varepsilon}, \text{ in } Q, \tag{1}$$

$$u_\varepsilon(t, 1) = 0, \text{ with } t \in (0, T), \tag{2}$$

$$u_\varepsilon(t, 0) = 0, \text{if } \alpha \in (0, 1), \text{ with } t \in (0, T), \tag{3}$$

$$(x^\alpha u_{\varepsilon x})(t, 0) = 0, \text{if } \alpha \in [1, 2), \text{ with } t \in (0, T), \tag{4}$$

$$u_\varepsilon(0, x) = u_0(x) \text{ and } u_\varepsilon(T, x) = 0, \text{with } x \in (0, 1), \tag{5}$$

with the following property: $(u_\varepsilon, h_\varepsilon)_{\varepsilon\in(0,1)}$ converges to (u, g), as $\varepsilon \to 0$, in a suitable sense, where (u, g) solves the boundary null controllability problem

$$u_t - (x^\alpha u_x)_x = 0, \text{ in } Q, \tag{6}$$

$$u(t, 1) = g(t), \text{ with } t \in (0, T), \tag{7}$$

$$u(t, 0) = 0, \text{if } \alpha \in (0, 1), \text{ with } t \in (0, T), \tag{8}$$

$$(x^\alpha u_x)(t,0) = 0, \text{if } \alpha \in [1,2), \text{ with } t \in (0,T), \tag{9}$$

$$u_(0,x) = u_0(x) \text{ and } u(T,x) = 0, \text{with } x \in (0,1), \tag{10}$$

where u_0 is an initial data taken in the weighted Sobolev space H^1_α, close related to α. In other words, we build a family of distributed state-control pairs for the degenerate parabolic equation (1)-(5), identifying its asymptotic behavior. To be more precise, this family provides a limiting pair which, in some sense, solves the boundary null controllability described in (6)-(10). In this investigation, we not only consider the **weakly degenerate case**, when $\alpha \in (0,1)$, but also the **strongly degenerate** one, when $\alpha \in [1,2)$ (see [1]).

2 Main Result

Finally, we are ready to state our main theorem, which rigorously clarifies what is said in the short introduction:

Theorem 2.1. *Consider $T > 0$ and $u_0 \in H^1_\alpha$. Then, for each $\varepsilon \in (0,1)$, there exist*

$$u_\varepsilon \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_\alpha) \cap C^0([0,T];H^1_\alpha) \text{ and } h_\varepsilon \in L^2(Q_\varepsilon)$$

such that:

(a) $(u_\varepsilon, h_\varepsilon)$ solves the distributed null controllability problem (1)-(5), in the sense of transposition;

(b) $u_\varepsilon \rightharpoonup u$, weakly in $L^2(Q)$, and $h_\varepsilon \rightharpoonup g$, weakly in $L^2(0,T;H^2_\alpha)$, as $\varepsilon \to 0$. Moreover,

$$u \in C^0([0,T];(H^1_\alpha)') \cap L^2(Q) \text{ and } g \in L^2(0,T),$$

with (u,g) solving the boundary null controllability problem (6)-(10), in the sense transposition.

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XIX ENAMA - Novembro 2026 *145–146*

INTERMEDIATE ASYMPTOTICS AND SHARP TRANSITION RATES FOR THE VISCOUS BURGERS EQUATION

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Abstract

We study the one-dimensional viscous Burgers equation on $(0, L)$ with conservative (no-flux) boundary conditions. The problem has two explicit nonlinear attractors parametrized by the conserved mass M: the non-Gaussian self-similar profile f_M of the half-line and the non-constant stationary profile U_M of the interval. Our main result is an explicit one-parameter family $\{\Phi_c\}_{c>0}$, the $L \to \infty$ limit of the rescaled solution at the diffusive time $t = cL^2$, which joins them: $\Phi_c \to f_M$ as $c \to 0$ and $\Phi_c \to U_M$ as $c \to \infty$. We determine the exact rate $e^{-1/c}$ of the first degeneration and establish a matching lower bound, so that the exponent $\gamma = 1$ is optimal.

1 Introduction

The large-time behavior of diffusive equations depends crucially on the geometry of the domain: on unbounded domains solutions spread to self-similar profiles, while on bounded domains confinement drives them to a stationary state. When the domain is large but finite the two mechanisms compete, and the diffusive time $t \sim L^2$ separates the regimes. The nature of the transition at this scale is the question we address.

We study the viscous Burgers equation with conservative boundary conditions,

$$u_t - u_{xx} = (u^2)_x \ \text{ in } (0,L)\times(0,\infty), \qquad u_x(0,t) = -u^2(0,t), \quad u_x(L,t) = -u^2(L,t). \tag{1}$$

Written conservatively (1) reads $u_t - \partial_x(u_x + u^2) = 0$, so the flux $-(u_x + u^2)$ vanishes at the endpoints and the mass $M = \int_0^L u\,dx$ is conserved. The main analytical tool is the Hopf–Cole transformation mapping u to $w = \exp\big(\int_0^x u\,dy\big)$, which solves the heat equation with Dirichlet data $w(0,t) = 1$, $w(L,t) = e^M$. Related results appear in [2, 3].

The conserved mass parametrizes both limit problems, whose attractors f_M and U_M are recalled below; the transient joining them lasts $t \sim L^2$ and is an instance of Barenblatt's intermediate asymptotics.

2 Main Results

Let $u_0 \in L^1(0, L)$ have mass M. The bounded-interval problem is well posed and preserves mass, and its unique stationary solution is

$$U_M(x) = \frac{e^M - 1}{x(e^M - 1) + L}. \tag{1}$$

Convergence to U_M is exponential, at the rate of the first Dirichlet eigenvalue $\lambda_1 = \pi^2/L^2$. On the half-line, self-similar solutions $u_M(x,t) = t^{-1/2} f_M(x/\sqrt{t})$ are governed by the explicit profile

$$f_M(\xi) = \frac{(e^M - 1)\, b(\xi)}{1 + (e^M - 1)\displaystyle\int_0^\xi b}, \qquad b(\xi) = \pi^{-1/2} e^{-\xi^2/4}. \tag{2}$$

Rescaling by $y = x/L$, $\tau = t/L^2$, the Hopf–Cole variable solves the heat equation on $(0,1)$ with constant Dirichlet data; let φ be its limit profile and $\Psi = 1 + (e^M - 1)\varphi$. Using the diffusive variable $\xi = x/\sqrt{t}$, we define the transition family.

Definition 2.1. *For $c > 0$ and $0 \le \xi < c^{-1/2}$,*

$$\Phi_c(\xi) := \sqrt{c}\,\frac{\Psi_y(\xi\sqrt{c}, c)}{\Psi(\xi\sqrt{c}, c)}. \tag{3}$$

The next result shows that $\{\Phi_c\}_{c>0}$ is the $L \to \infty$ limit of the rescaled solution at diffusive times $t = cL^2$, and joins the two attractors.

Theorem 2.1. *Assume $\operatorname{supp} u_0 \subset [0, R_0]$ and fix $c > 0$, $k \in \mathbb{N}_0$ and $R \in (0, c^{-1/2})$. There is $C > 0$ such that, for $t = cL^2$ and all $L \ge R_0$,*

$$\big\|\sqrt{t}\,u^L(\cdot\sqrt{t}, cL^2) - \Phi_c\big\|_{C^k([0,R])} \le \frac{C}{L}. \tag{4}$$

Moreover $\Phi_c \to f_M$ as $c \to 0^+$ and $\Phi_c \to U_M$-type profile as $c \to \infty$.

Proposition 2.1. *$\{\Phi_c\}_{c>0}$ connects the profiles f_M and U_M and no others, and its existence identifies $t \sim L^2$ as the transition scale.*

The first degeneration $\Phi_c \to f_M$ occurs at a sharp rate, driven by the image charge at $x = 2L$ generated by the boundary, whose Gaussian cost is $e^{-1/c}$.

Theorem 2.2. *Assume $\operatorname{supp} u_0 \subset [0, R_0]$ with $M \neq 0$ and fix $R > 0$. There are $c_1, \kappa_\pm > 0$ such that, for $0 < c \le c_1$ and every $k \in \mathbb{N}_0$,*

$$\big\|\Phi_c - f_M\big\|_{C^k([0,R])} \le C_{k,\varepsilon}\, e^{-(1-\varepsilon)/c}, \qquad \big|\Phi_c(0) - f_M(0)\big| \ge \kappa_-\, e^{-1/c}, \tag{5}$$

so that $\lim_{c\to 0^+} -c \ln\big|\Phi_c(0) - f_M(0)\big| = 1$.

Numerical experiments corroborate all the theoretical results. This is a summary based on [1].

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XIX ENAMA - Novembro 2026 *147–148*

BLOW-UP FOR A SEMILINEAR TRICOMI-TYPE EQUATION WITH SCALE-INVARIANT MASS IN THE OSCILLATORY REGIME

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Abstract

We investigate finite-time blow-up for a semilinear Tricomi-type equation with decreasing propagation speed and scale-invariant mass. While most available results concern the nonoscillatory regime, we treat the complementary oscillatory case. We prove nonexistence of global finite-propagation weak solutions in a Strauss-type range and derive a lifespan upper bound. The proof combines a positive adjoint mode with tests localized according to the logarithmic oscillations of a Cauchy-Euler profile.

1 Introduction

Let $n \geq 1$, $-1 < \ell < 0$, $\mu^2 > 1/4$, $\alpha \geq 0$, and $p > 1$. We consider the semilinear Cauchy problem

$$\begin{cases} u_{tt} - (1+t)^{2\ell}\Delta u + \dfrac{\mu^2}{(1+t)^2}u = |u|^p, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = \varepsilon f(x), \qquad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\varepsilon > 0$ and the initial data are smooth, nonnegative, compactly supported, and not both zero. The factor $(1+t)^\ell$ represents the time-dependent propagation speed, which, since $-1 < \ell < 0$, is positive and decreases to zero as $t \to \infty$. The zeroth-order term $\mu^2(1+t)^{-2}u$ is a time-dependent mass term in the Klein-Gordon sense.

The main result below shows that the sign of the weighted Strauss polynomial

$$\gamma(n,\ell;p) := \left(n - 1 + \frac{\ell}{1+\ell}\right)p^2 - \left(n + 1 - \frac{3\ell}{1+\ell}\right)p - 2$$

governs the nonexistence region: if $\gamma_\alpha(n,\ell;p) < 0$, then every nontrivial finite-propagation weak solution with nonnegative compactly supported initial data blows up in finite time, and hence no global solution exists. This result is consistent with the classical Strauss conjecture for the semilinear wave equation; see [1].

The spatially homogeneous linear dynamics associated with (1) is governed by the Cauchy-Euler equation

$$h''(t) + \frac{\mu^2}{(1+t)^2}h(t) = 0. \tag{2}$$

The solutions of this ODE are determined by the indicial discriminant by $\delta := 1 - 4\mu^2$. If $\delta \geq 0$, the indicial roots are real and the solutions are power-like with globally controlled sign. Most of the literature on scale-invariant wave equations deals with this regime, because the Cauchy-Euler solutions can be chosen to provide globally positive weights or test functions, preserve the monotonicity of suitable space averages, and allow the direct application of standard Kato-type, iteration, and test-function arguments, see [1, 2].

In our case, $\mu^2 > 1/4$, and therefore $\delta < 0$. The a real solution of (2) is

$$h(t) = (1+t)^{1/2}\cos\big(\omega\log(1+t)\big), \qquad \omega = \sqrt{\mu^2 - \frac{1}{4}}$$

Consequently, every nontrivial real solution is oscillatory and changes sign infinitely many times, which prevents its direct use as a globally positive test function. Our strategy is to retain the temporal geometry encoded by h while recovering positivity on carefully selected intervals $I_j \subseteq [0,T)$ on which h is positive. On each such interval, the oscillatory test function yields an upper estimate for the relevant integral involving $|u|^p$. We then compare these upper estimates with lower bounds for the same quantity obtained by means of the classical positive adjoint function associated with the problem. The incompatibility between the resulting upper and lower bounds ultimately provides the mechanism leading to the nonexistence result.

2 Main Result

Theorem 2.1. *Assume the above hypotheses and let $f, g \in C_c^\infty(\mathbb{R}^n)$ be nonnegative and not both identically zero. If $\gamma(n,\ell;p) < 0$, then (1) has no global finite-propagation weak solution for any $\varepsilon > 0$. Moreover, there exist $\varepsilon_0, C > 0$, independent of ε, such that, for $0 < \varepsilon \le \varepsilon_0$, the lifespan satisfies*

$$T_\varepsilon \le C\varepsilon^{\frac{2p(p-1)}{(1+\ell)\gamma(n,\ell;p)}}. \tag{1}$$

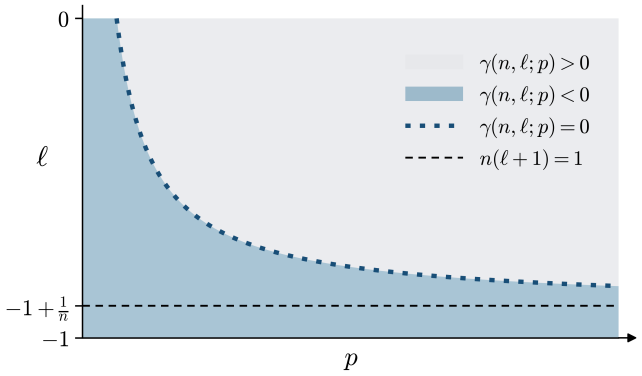


Figure 1: Blow-up region

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XIX ENAMA - Novembro 2026 *149–150*

UPPER AND LOWER ESTIMATES FOR DERIVATIVES OF SOLUTIONS TO A CLASS OF ADVECTION-DIFFUSION EQUATIONS

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Abstract

We study upper and lower estimates for spatial derivatives of solutions to a class of conservative advection-diffusion equations. By combining energy methods with a bootstrap argument, we derive decay estimates for the derivatives of sufficiently regular solutions. In particular, we obtain L^2-decay estimates for first-order derivatives, with rates depending on the nonlinear exponent of the advective term. The results contribute to a better understanding of the asymptotic behavior of nonlinear advection-diffusion equations.

1 Introduction

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In this work, we consider the nonlinear advection-diffusion equation

$$u_t + \operatorname{div}\left(b(x,t,u)|u|^k u\right) = \mu(t)\Delta u, \tag{1}$$

where $x \in \mathbb{R}^n$, $t > 0$, $k \geq 0$, with initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We assume that the function $\mu \in C^0([0,\infty))$ is positive, and that the advective field b is continuously differentiable, bounded, i.e. $|b(x,t,u)| \leq B(t)$ for some $B \in C^0([0,\infty))$, and satisfies

$$\operatorname{div} b(x,t,u) \geq 0. \tag{2}$$

Equation (1) can be viewed as a natural generalization of the classical viscous Burgers equation. More precisely, the quadratic Burgers flux is replaced by the nonlinear flux $b(x,t,u)|u|^k u$, allowing the advective field to depend explicitly on the spatial variable, time, and the solution itself. This additional flexibility considerably enlarges the class of nonlinear advection–diffusion equations and introduces new analytical challenges in the study of their asymptotic behavior.

Equations combining nonlinear advection and diffusion have been extensively investigated due to their mathematical interest and numerous applications. Several results concerning the global existence and decay of solutions are available under different assumptions on the advective field, see [1, 2, 3, 4, 5]. However, while decay estimates for the solutions themselves are now relatively well understood, considerably less is known about the asymptotic behavior of their spatial derivatives, especially for equations involving general nonlinear fluxes.

The purpose of this work is to investigate the long-term behavior of bounded solutions to (1). Our approach combines L^p-decay estimates, energy methods, and a bootstrap argument to derive decay estimates for spatial derivatives, providing new information on the regularity and long-time behavior of solutions.

2 Main Results

In this section, we derive several important properties of the solution u. Among them, we establish decay estimates for u in the L^p norm, which will play a fundamental role in deriving decay rates for its spatial derivatives.

Proposition 2.1. *Let $u(\cdot,t) \in C^0([0,T_*],L^1(\mathbb{R}^n)) \cap L^\infty_{loc}([0,T_*],L^\infty(\mathbb{R}^n))$ be any given solution of problem (1) under hypothesis (2). Then, for $1 \leq p$,*

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \leq \|u(\cdot,0)\|_{L^p(\mathbb{R}^n)},$$

where $[0,T_*]$ is the maximal interval of existence of the solution.

Furthermore, for all $k \geq 0$, we prove that for all $1 \leq p < \infty$, the following result holds

Proposition 2.2. *Let $u(\cdot,t) \in C^0([0,T_*],L^1(\mathbb{R}^n)) \cap L^\infty_{loc}([0,T_*],L^\infty(\mathbb{R}^n))$ be any given solution of problem (1) under hypothesis (2). Then,*

$$\|u(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^n)} \leq C(n,p)\|u_0\|^p_{L^p(\mathbb{R}^n)} t^{-(1+\frac{n}{2})} \int_0^t \mu(\tau)^{-\frac{n}{2}}. \tag{1}$$

And, for $p = \infty$,

Proposition 2.3. *Let $u(\cdot,t) \in C^0([0,T_*],L^1(\mathbb{R}^n)) \cap L^\infty_{loc}([0,T_*],L^\infty(\mathbb{R}^n))$ be any given solution of problem (1) under hypothesis (2). Then,*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq C(n,p)\Big(\int_0^t \mu(\tau)^{-\frac{n}{2}}\Big)^{\frac{1}{p}} \|u_0\|_{L^p(\mathbb{R}^n)} t^{-\frac{1}{p}\left(1+\frac{n}{2}\right)}. \tag{2}$$

Using estimates (1) for $p = 1$ and (2) combined with energy methods and a bootstrap argument, it is possible to obtain decay rates for the L^2-norm of the spatial derivatives of the solution to problem (1). Thus, we obtain one of the key results of our work.

Theorem 2.1. *Let $u(\cdot,t)$ be any given solution to problem (1) under hypotheses (2). Then*

$$\|Du(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}-min\{1,nk\}}, \quad \forall t > t_* > 0.$$

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LAMÉ SYSTEM WITH NONLINEAR VECTOR FIELD UNDER CRITICAL EXPONENT

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Abstract

In this work, we investigate the existence of a compact global attractor that has finite fractal dimension for the Lamé system. The decomposition of the nonlinearity into a coupled and a decoupled component is a well-established approach in the mathematical literature. Our objective is to propose an alternative method capable of handling the critical case without requiring this structural assumption on the nonlinearity, that is, we consider the general case for the nonlinear vector fields under critical exponents.

1 The problem

Let us consider the following problem

$$\begin{cases} \mathbf{u}_{tt} - \Delta_e \mathbf{u} + \mathbf{f}(\mathbf{u}) + \alpha \mathbf{u}_t = \mathbf{h} \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ \mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}_t(0) = \mathbf{u}_1 \quad \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{u} = (u_1, u_2, u_3)$ is the unknown vector solution depending on (x, t), with $x = (x_1, x_2, x_3) \in \Omega$ and $t \geq 0$, $\mathbf{f}(\mathbf{u})$ is a vector-valued nonlinear perturbation, $\mathbf{h} = \mathbf{h}(x)$ is an external force in $L^2(\Omega)$, α is the damping coefficient, and Δ_e stands for the elastic Lamé operator given by

$$\Delta_e \mathbf{u} = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \text{div} \mathbf{u}. \tag{2}$$

2 Main Results

Assumption 1. Let us assume $\mathbf{h} \in \mathbf{L}^2(\Omega)$, $\mu > 0$, $\lambda + \mu \geq 0$ and $\alpha > 0$.

Concerning the nonlinear operator $\mathbf{f}(\mathbf{u}) = (\mathbf{f_1}(\mathbf{u}), \mathbf{f_2}(\mathbf{u}), \mathbf{f_3}(\mathbf{u}))$, we assume the following growth condition in critical scenarios.

Assumption 2. Let us assume that there exist functions $\mathbf{g} = (g_1, g_2, g_3) \in \mathbf{C}^1(\mathbb{R}^3)$, $G \in C^2(\mathbb{R}^3)$ and $\gamma_i \in C^2(\mathbb{R})$, $i = 1, 2, 3$, such that the vector field $\mathbf{f} = (f_1, f_2, f_3)$ satisfies:

$$f_i(u_1, u_2, u_3) = g_i(u_1, u_2, u_3) + \gamma_i(u_i), \quad i = 1, 2, 3,$$

$$f_i(0) = g_i(0) = \gamma_i(0) = 0, \quad i = 1, 2, 3,$$

$$\mathbf{g} = (g_1, g_2, g_3) = \nabla G.$$

In addition, there exist constants $M, m_f \geq 0$ such that

$$f(u) \cdot u - G(u) - \sum_{i=1}^{3} \int_0^{u_i} \gamma_i(s)\, ds \geq -M|u|^2 - m_f, \quad \forall u \in \mathbb{R}^3, \tag{1}$$

$$G(u) + \sum_{i=1}^{3} \int_0^{u_i} \gamma_i(s)\, ds \geq -M|u|^2 - m_f, \quad \forall u \in \mathbb{R}^3, \tag{2}$$

with $0 \leq M < \frac{\mu\lambda_1}{2}$, where $\lambda_1 > 0$ denotes the first eigenvalue of the Laplacian operator $-\Delta$. With respect to functions g_i and γ_i, $i = 1, 2, 3$, we assume:

- g fulfills the growth restriction: there exist $M_g > 0$ such that, for $i = 1, 2, 3$,

$$|\nabla g_i(u)| \leq M_g \left(1 + |u_1|^{p-1} + |u_2|^{p-1} + |u_3|^{p-1}\right), \quad \forall u = (u_1, u_2, u_3) \in \mathbb{R}^3, \tag{3}$$

  for $1 \leq p \leq 3$.

- For each $i = 1, 2, 3$, γ_i fulfills the critical growth restriction: there exists a constant $c_h > 0$ such that

$$|\gamma_i'(x)| \leq c_\gamma(1 + |x|^2), \quad \forall x \in \mathbb{R}, \quad i = 1, 2, 3. \tag{4}$$

By denoting $\mathbf{v} = \mathbf{u}_t$ and $\mathbf{U}(t) = (\mathbf{u}(t), \mathbf{v}(t))$, $t \geq 0$, we rewrite problem (1) in the following abstract Cauchy problem

$$\begin{cases} \mathbf{U}_t = \Delta_{\mathcal{E}}\mathbf{U} + \mathbf{F}(\mathbf{U}), \quad t > 0, \\ \mathbf{U}(0) = (\mathbf{u}_0, \mathbf{u}_1) := \mathbf{U}_0, \end{cases} \tag{5}$$

with linear $\Delta_{\mathcal{E}} : D(\Delta_{\mathcal{E}}) \subset \mathcal{H} \to \mathcal{H}$ and nonlinear $\mathbf{F} : \mathcal{H} \to \mathcal{H}$ operators given by

$$\Delta_{\mathcal{E}}\mathbf{U} = (\mathbf{v}, \Delta_e\mathbf{u} - \alpha\mathbf{v}) \quad \text{and} \quad \mathbf{F}(\mathbf{U}) = (0, \mathbf{h} - \mathbf{f}(\mathbf{u})),$$

with $\mathcal{H} = [H_0^1(\Omega)]^3 \times [L^2(\Omega)]^3$ and $D(\Delta_{\mathcal{E}}) = [(H^2(\Omega) \cap H_0^1(\Omega))]^3 \times [H_0^1(\Omega)]^3$. Under Assumptions 1 and 2, one can show that problem (5) is well-posed on $\mathcal{H}$. Then, we can generate the dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (1) via mild solutions $\mathbf{U}(t) = (\mathbf{u}(t), \mathbf{u}_t(t))$, $t \geq 0$, of (5), namely,

$$S(t)(\mathbf{u}_0, \mathbf{u}_1) = (\mathbf{u}(t), \mathbf{u}_t(t)), \quad (\mathbf{u}_0, \mathbf{u}_1) \in \mathcal{H}. \tag{6}$$

In [1, 2] the authors consider $p < 3$ (subcritical growth condition) in (3) or $g_i = 0$ for $i = 1, 2, 3$ in (3). Here, our main goal is to explore the critical case $p = 3$ in (3), that is, critical growth condition for the vector field $\mathbf{g} = (g_1, g_2, g_3)$. More precisely, we have:

Theorem 2.1. *Under the Assumptions 1 and 2 with $p = 3$, the dynamical system $(\mathcal{H}, S(t))$ given by (6) possesses a unique compact global attractor $\mathcal{A}$ with finite fractal dimension in $\mathcal{H}$ ($\dim_{\mathcal{H}}^f \mathcal{A} < \infty$). Moreover, the attractor $\mathcal{A}$ is bounded in $D(\Delta_{\mathcal{E}})$.*

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XIX ENAMA - Novembro 2026 *153–154*

REGULARITY FOR SOLUTIONS OF DEGENERATE WAVE EQUATION WITH NORMAL DERIVATIVE IN L^2

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Abstract

For the wave equation in $\Omega \subset \mathbb{R}^n$, a solution by transposition v with initial data $(v^0, v^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and normal derivative $\partial_\nu v \in L^2(\partial\Omega)$ is known to be a weak solution corresponding to $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$. While this hidden regularity result fails in general for hyperbolic equations with non-smooth coefficients, we establish an analogous result for the one-dimensional degenerate wave equation.

1 Introduction

In [1], a result was established for the degenerate one-dimensional wave equation

$$u_{tt} - (x^\alpha u_x)_x = h\chi_\omega, \quad (t,x) \in (0,T) \times (0,1), \tag{1}$$

concerning the asymptotic behavior of a family of distributed control problems, where the control domain ω is localized in a neighborhood of the boundary $x = 1$ that shrinks to $x = 1$. The rigorous proof of this result relies on obtaining an additional regularity property for solutions by transposition to the homogeneous adjoint equation, specifically when the normal derivative belongs to $L^2(0,T)$.

For the standard non-degenerate wave equation in $\Omega \subset \mathbb{R}^n$, this property is noted by Lions in Remark 2.4 of [3]. However, due to the presence of the degenerate coefficient in (3), this regularity result does not follow as a direct consequence.

The results of [1] were first presented at the XVII ENAMA. Following peer review, it became clear that the regularity property for the degenerate equation required a more rigorous proof. Here, we present the complete and detailed justification.

Fix $\alpha \in (0,2)$ and $T > 0$. Given $(v^0, v^1) \in L^2(0,1) \times H_\alpha^{-1}(0,1)$, let v be the solution by transposition of the problem

$$\begin{cases} v_{tt} - (x^\alpha v_x)_x = 0, & (t,x) \in (0,T) \times (0,1), \\ \mathcal{B}v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(0,\cdot) = v^0, \ v_t(0,\cdot) = v^1. \end{cases} \tag{2}$$

where $\mathcal{B}u(t,0) = 0$ is the suitable boundary condition related to the degenerate operator at $x = 0$, defined by

$$\mathcal{B}v(t,0) = \begin{cases} v(t,0), & \text{if } \alpha \in (0,1), \\ \lim\limits_{x \to 0+} (x^\alpha v_x)(t,x), & \text{if } \alpha \in [1,2), \end{cases}, t \in (0,T).$$

Proposition 1.1. *There exists $\widetilde{T}_\alpha \geq \frac{4}{2-\alpha}$ such that, for any $T > \widetilde{T}_\alpha$ and v be the solution by transposition of (2) with initial data $(v^0, v^1) \in L^2(0,1) \times H_\alpha^{-1}(0,1)$. If $v_x(\cdot,1) \in L^2(0,T)$, then $(v^0, v^1) \in H_\alpha^1(0,1) \times L^2(0,1)$.*

The proof relies on describing the solutions of (2) using the Fourier-Bessel series, as in [2]. The result follows from a suitable application of the Ingham inequality, which also provides the following definition to the time $\tilde{T}_\alpha$:

$$\widetilde{T}_\alpha = \begin{cases} \frac{4\pi}{(2-\alpha)(j_{\nu,2}-j_{\nu,1})}, & 0 \leq \alpha \leq \frac{4}{3}, \\ \frac{4}{2-\alpha}, & \frac{4}{3} < \alpha < 2, \end{cases} \tag{3}$$

where $\nu = \frac{|1-\alpha|}{2-\alpha}$ and $j_{\nu,1}$, $j_{\nu,2}$ are the first and second zeros of the Bessel function J_ν.

We know that $T_\alpha = \frac{4}{2-\alpha}$ is the minimum time where the equation (3) can be controlled. We can visualize how close $\tilde{T}_\alpha$ is to T_α, when $\alpha \in \left[0, \frac{4}{3}\right]$, by the following plot of the ratio between them

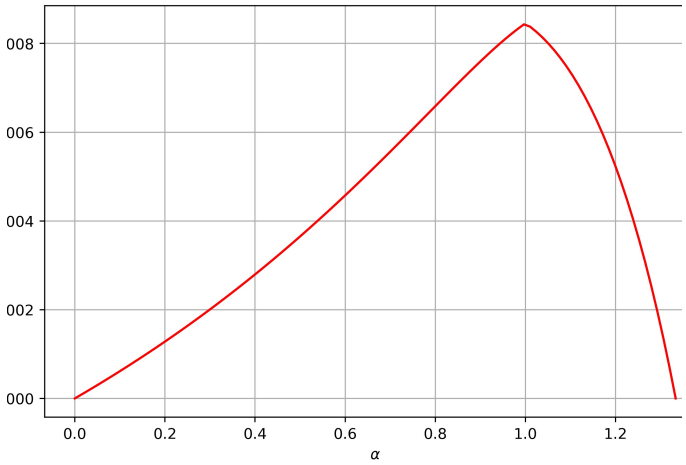


Figure 1: ratio $r_\alpha = \frac{\widetilde{T}_\alpha}{T_\alpha}$, when $0 \leq \alpha \leq \frac{4}{3}$.

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XIX ENAMA - Novembro 2026 *155–156*

PSEUDO-DIFFERENTIAL HOMOGENIZATION PROCEDURE IN THE COMPOSITE MATERIALS.

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Abstract

In the present work we are considering the homogenization procedure of the Cauchy problem for the equations of the elasticity. This procedure is based on the operator separation of variables. This method gives the new incites for the homogenization of the elasticity equations and provides new way of the homogenization and constructing the homogenized equation. This approach can be extended on situations with quasi-periodic coefficients.

1 Introduction

The homogenization problem in the composite materials is very important problem in the applications and in the fundamental studies [1]. We are considering composite material similar to [2]. The composite material is constructed via periodic repetition of the two materials united in the cell

$$Y^\varepsilon = \{x_2 \in \mathbb{R},\, x_1 \in [0,\, \varepsilon l_1]\}, \quad y_1 = \frac{x_1}{\varepsilon} \in [0,\, l_1].$$

This unit cell describes the composite material consisting of two different elements which are periodically repeated. We introduce the coefficient $\gamma \in (0,\, 1)$ which describes the width of each material in the unit cell - material A: $y_1 \in [0,\, \gamma l_1)$ and material B: $y_1 \in (\gamma l_1,\, l_1]$. Here l_1 is the length of the cell along x_1 axis and parameter $\varepsilon \ll 1$ is a small parameter which is the ratio of length l_1 to the distance of the wave propagation in the composite material.

We pose the following Cauchy problem

$$\nabla_x \cdot \boldsymbol{\sigma}^\varepsilon(x,\, t) - \rho(\frac{x_1}{\varepsilon})\frac{\partial^2}{\partial t^2}\mathbf{u}^\varepsilon(x,\, t) = 0, \quad x = (x_1,\, x_2) \in \mathbb{R}^2, \tag{1}$$

$$\mathbf{u}^\varepsilon(x,\, 0) = \mathbf{f}(x), \quad \frac{\partial \mathbf{u}^\varepsilon}{\partial t}(x,\, 0) = \mathbf{g}(x). \tag{2}$$

Here $\mathbf{u}^\varepsilon(x,\, t) = (u_1(x,\, t),\, u_2(x,\, t))$ and $\boldsymbol{\sigma}^\varepsilon(x,\, t) = C(x_1/\varepsilon) : (\nabla \otimes \mathbf{u}^\varepsilon + (\nabla \otimes \mathbf{u}^\varepsilon)^T)/2$. Functions $\mathbf{f}(x)$ and $\mathbf{g}(x)$ are the given vector-functions. Operation " : " denotes the double contraction.

The tensor $C(y_1) = C_{ijkl}(y_1)$ describes the properties of the material and it is the symmetric and completely positive tensor of the fourth order. Function $\rho(y_1)$ describes the density of the material.

During the presence of two materials in the unit cell we introduce additional conditions on the interface points

$$[\![\mathbf{u}^\varepsilon(x,\, t)]\!] = 0, \quad [\![\boldsymbol{\sigma}^\varepsilon(x,\, t) \cdot \mathbf{n}]\!] = 0, \quad x \in S^\varepsilon \equiv \bigcup_{n \in \mathbb{Z}} \left\{x = (x_1, x_2) \in \mathbb{R}^2 :\ x_1 = \varepsilon(n + \gamma) l_1\right\}. \tag{3}$$

Here $\mathbf{n}$ is the outward normal vector along the positive direction of x_1. The double brackets $[\![\cdot]\!]$ denotes the jumps of the functions in the interface points.

2 Main Results

We introduce the tensors $C^1(y_1)$ and $C_1(y_1)$ of the 3-rd order and the symmetric matrix $\mathcal{C}(y_1)$

$$C^1(y_1) = C_{i1kl}(y_1), \quad C_1(y_1) = C_{ijk1}(y_1), \quad \mathcal{C}(y_1) = \{C_{i1k1}(y_1)\} = \begin{pmatrix} C_{1111}(y_1) & C_{1121}(y_1) \\ C_{2111}(y_1) & C_{2121}(y_1) \end{pmatrix}. \tag{1}$$

Let the brackets $\langle\cdot\rangle$ denote the mean value of the corresponding function on $y_1 \in [0,\, l_1]$.

Theorem 2.1. *The tensor*

$$\overline{C} = \langle C(y_1)\rangle - \langle C_1(y_1)\mathcal{C}^{-1}(y_1)C^1(y_1)\rangle + \langle C_1(y_1)\mathcal{C}^{-1}(y_1)\rangle\langle\mathcal{C}^{-1}(y_1)\rangle^{-1}\langle\mathcal{C}^{-1}(y_1)C^1(y_1)\rangle. \tag{2}$$

is positive-definite and completely symmetrical conditions, similar to the tensor $C(y_1)$.

Definition 2.1. *We call the equation*

$$\nabla_x \cdot \boldsymbol{\sigma}^0(x,\, t) - \langle\rho(y_1)\rangle\frac{\partial^2}{\partial t^2}\mathbf{u}^0(x,\, t) = 0,\ \boldsymbol{\sigma}^0(x,\, t) = \frac{1}{2}\overline{C} : (\nabla \otimes \mathbf{u}^0 + (\nabla \otimes \mathbf{u}^0)^T). \tag{3}$$

the homogenized equation.

Theorem 2.2 (Homogenization theorem). *Let function $\mathbf{u}^0(x,\, t)$ be a solution of the equation (3), and let the tensor*

$$\mathcal{A}(y_1) = \mathcal{C}^{-1}(y_1)\Big(C^1(y_1) - \langle\mathcal{C}^{-1}(y_1)\rangle^{-1}\langle\mathcal{C}^{-1}(y_1)C^1(y_1)\Big).$$

There is a bounded function $\mathbf{u}_2(x,\, x_1/\varepsilon,\, t)$ such that the function

$$\mathbf{u}_{as}(x,\, t) = \mathbf{u}^0(x,\, t) - \varepsilon\left(\int\limits_0^{x_1/\varepsilon} \mathcal{A}(\eta)d\eta\right) : (\nabla_x \otimes \mathbf{u}^0(x,\, t)) + \varepsilon^2\mathbf{u}_2(x,\, t) \tag{4}$$

satisfies the initial equation (1) and initial conditions (2) with the residue $O(\varepsilon)$.

The proof of the Theorem 2.2 is based on the operator separation of slow and fast variables [3] and the existence theorems from [1]. The big advantage of this approach that this method can be extended on the quasi-periodic structures, contrary on the method presented in [2].

This is the joined work with R. Rodríguez-Ramos, Federal University of Fluminense and University of Havana.

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XIX ENAMA - Novembro 2026 *157–158*

RAINFALL-TRIGGERED SLOPE LIQUEFACTION MODELING IN JUIZ DE FORA VIA HYPERBOLIC CONSERVATION LAWS

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Abstract

In this work, we model rainwater-induced slope liquefaction using hyperbolic conservation laws in a one-dimensional, two-phase setting. We find a solution for the boundary Riemann problem using the Oleynik entropy condition. As an application, motivated by recent heavy rainfalls in Juiz de Fora, we use the slope inclination (θ) and Brooks-Corey retention parameters for Zona da Mata Mineira soils based on the *HYBRAS* database. Our results indicate that high-permeability Red-Yellow Latosol produces a pure shock wave, leading to abrupt saturation and rapid slope failure. In contrast, Haplic Cambisol and Yellow Latosol exhibit a composite rarefaction-shock wave structure. The analytical framework provides exact saturation wave speeds and slope-failure times, which are a potentially useful tools for geotechnical early-warning systems.

1 Physical formulation and two-phase Darcy Flow

The soil was modeled as a porous medium with a two-phase flow of water (w) and air (a). Assuming constant atmospheric pressure at the outflow of the porous medium and neglecting the capillary pressure gradient ($\nabla p_c = 0$), we can treat the pressure as a single variable ($p_\alpha = p$). The model is:

- Darcy Velocity for phase α: $u_\alpha = -\lambda_\alpha(S_w)\cdot(\nabla p_\alpha - \rho_\alpha g\sin\theta)$, where $\lambda_\alpha(S_w) = \big(k\, k_{r\alpha}(S_w)\big)/\mu_\alpha$ is a phase mobility, k is the absolute permeability, $k_{r\alpha}$ is the relative permeability, μ_α is the dynamic viscosity, ρ_α is the density, g is the gravitational acceleration, and θ is the slope inclination.

- Combining total flow velocity $u = u_w + u_g$ and isolating water phase velocity:

$$u_w = [u + \lambda_g(\rho_w - \rho_g)g\sin\theta]\cdot f_w(S_w), \tag{1}$$

  where $f_w(S_w) \equiv \lambda_w/(\lambda_w + \lambda_g)$ is the fractional flow function of the water phase.

- Brooks-Corey relative permeability model is

$$k_{rw}(S) = k^0_{rw}S^{n_w}, \quad k_{rg}(S) = k^0_{rg}(1-S)^{n_g} \quad \text{with } n_w = 3 + \frac{2}{\lambda},\ n_g = 1 + \frac{2}{\lambda}, \tag{2}$$

  expressed in terms of effective saturation

$$S = \frac{S_w - S_{wc}}{1 - S_{wc} - S_{rg}} \in [0,1].$$

2 The Riemann Problem

The flow dynamics, along the slope, is governed by the mass conservation law of the water phase:

$$\frac{\partial S}{\partial t} + \frac{\partial F(S)}{\partial x} = 0, \quad \text{with} \quad F(S) = \frac{u_w}{\phi(1 - S_{wc} - S_{rg})}. \tag{3}$$

The initial condition corresponds to the left state S_l and the right state $S_r = 0$ for $x > 0$, since we assume the slope is initially completely dry. We show that the solution presents a bifurcation between shock and shock, followed by a rarefaction, which happens for

$$S_l = min\left(1, \left(\frac{u\mu_w}{kk_{rw}^0(\rho_w - \rho_g)gsin\theta}\right)^{\frac{1}{n_w}}\right), \tag{4}$$

where u is the constant flow velocity along the porous medium. In this case, the shock propagation velocity is determined by the Rankine-Hugoniot condition:

$$s = \frac{F(S_l) - F(S_r)}{S_l - S_r}. \tag{5}$$

3 Oleynik's Entropy Conditions

Given the sigmoidal structure of the flux function $F(S)$, the physically admissible solution must follow the Oleynik's Entropy Conditions. In that case, the solution is split into two cenarios:

(a) Pure Shock: A discontinuous jump in soil saturation.

(b) Rarefaction + Shock: A continuous saturation gradient followed by a sharp shock front, according to the Buckley-Leverett solution.

4 Application results

The model was tested on three tropical soil profiles typical of the Zona da Mata from the *HYBRAS* database: Haplic Cambisol, Yellow Latosol, and Red-Yellow Latosol. Adopting a slope angle $\theta = 30°$ and rainfall rate $u = 10^{-6}$ m/s:

Table 1: Permeability, flow regimes, and slope failure behavior of soils.

Soil Profile	**Permeability k (m^2)**	**Flow Regime**	**Failure Behavior**
Red-Yellow Latosol	4.21×10^{-12}	Pure Shock	High permeability, abrupt saturation front without a transition zone, instantaneous slope failure.
Haplic Cambisol	1.03×10^{-12}	Rarefaction + Shock	Lower permeability, continuous transition zone (S_l to S_m) before the trailing shock wave.
Yellow Latosol	3.93×10^{-13}	Rarefaction + Shock	Gradual saturation growth prior to the shock front.

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UFJF - Universidade Federal de Juiz de Fora
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EXISTENCE AND CONTINUOUS DEPENDENCE OF THE LOCAL OR GLOBAL SOLUTION FOR NONHOMOGENEOUS HEAT EQUATIONS

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Abstract

In this work, we prove that the initial value problem associated with the nonhomogeneous heat equation on periodic Sobolev spaces has a local solution in $[0,T]$ with $T>0$, and the solution has a continuous dependence with respect to the initial data and the nonhomogeneous part of the problem. We do this intuitively using Fourier theory and introducing a C_o-semigroup inspired by the work of Iorio [4] and Ayala [1]. Also, we prove the uniqueness solution of the homogeneous and nonhomogeneous problems, using its dissipative property, inspired by the work of Iorio [4], Liu [5] and Ayala [2]. Moreover, by introducing a uniform external source, we obtain a global solution to the nonhomogeneous problem and its uniform decay. Finally, we generalize the results.

1 Introduction

First, we want to comment that from Theorem 3.1 in [3], we see that the homogeneous problem is globally well posed and, in addition to the inequality (3.2) in [3], we have a continuous dependence of the solution of the homogeneous problem. In this work, in Theorem 2.1 we will prove the existence and uniqueness of the local solution for the non-homogeneous problem and from inequality (3) we will get the continuous dependence of the solution with respect to the initial data and with respect to the non-homogeneous part. Thus, in both homogeneous and non-homogeneous cases, the estimatives are made from the explicit form of the solution, that is, by applying the Fourier transform to the respective equation. Another result in this work is about the dissipative property of the homogeneous problem and some estimates of it, using the differential calculus in H^s_{per}. This is included in Theorem 2.2 which we will develop. So, using Theorem 2.2, we deduce the results of continuous dependence and uniqueness of solution for both homogeneous and non-homogeneous problems, respectively; and the exponential stability of the semigroup. In Theorem 2.3 we obtain the existence of a global solution of (P^F) and its uniform decay. Finally, we give some properties of the differential operator, such as the spectral analysis, conclusions, and generalizations.

2 Main Results

We will prove that the nonhomogeneous problem (P^F) is well posed locally.

Theorem 2.1. *Let $\varphi \in H^s_{per}$, $s \in R$, $a>0$, $b>0$, $T>0$, $F \in C([0,T],H^s_{per})$ and $\{S(t)\}_{t\geq 0}$ the C_o-semigroup of contraction in H^s_{per}, introduced in Theorem 3.2 from [3], then*

1. *The function $u^F(t) := S(t)\varphi + \int_0^t S(t-\tau)F(\tau)\,d\tau$, $t\in[0,T]$ belong to $C([0,T],H^s_{per}) \cap C^1((0,T],H^{s-2}_{per})$*
2. *$u := u^F$ is the unique solution of*

$$(P^F) \quad \left| \begin{array}{l} u_t - b\,\partial_x^2 u + au = F\,, \quad F(t)\in H^{s-2}_{per} \\ u(0)=\varphi \end{array} \right. \tag{1}$$

with the derivative given by

$$\lim_{h\to 0} \left\| \frac{u(t+h)-u(t)}{h} - b\,\partial_x^2 u + au - F(t) \right\|_{s-2} = 0 \tag{2}$$

3. *Let $\varphi_j \in H^s_{per}$, $F_j \in C([0,T], H^s_{per})$ for $j = 1, 2$. Then the map $\varphi \longrightarrow u$ is continuous in the following sense. Let u_j be the corresponding solutions to initial data φ_j with non homogeneity F_j, respectively, then*

$$\|u_1(t) - u_2(t)\|_s \leq e^{-at}\|\varphi_1 - \varphi_2\|_s + \frac{1}{a}(1 - e^{-at})\|F_1 - F_2\|_{\infty,s}\,,\ t \in [0,T] \tag{3}$$

$$\begin{aligned}\|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-2} &\leq (b+a)e^{-at}\|\varphi_1 - \varphi_2\|_s + [(1 + \frac{b}{a})(1 - e^{-at}) + 1]\|F_1 - F_2\|_{\infty,s} \\ &\leq (b+a)\|\varphi_1 - \varphi_2\|_s + [2 + \frac{b}{a}]\|F_1 - F_2\|_{\infty,s} \end{aligned} \tag{4}$$

where we have used the notation: $\|g\|_{\infty,r} = \sup_{t\in[0.T]} \|g(t)\|_r\ ,\quad g \in C([0,T], H^r_{per})\,.$

Now, we study the dissipative property of the homogeneous problem

Theorem 2.2. *Let w the solution of (P^F) with $F = 0$ and initial data $\phi \in H^s_{per}$, then the following statements hold:*

1. $\partial_t\|w(t)\|^2_{s-2} = -2b\|\partial_x w(t)\|^2_{s-2} - 2a\|w(t)\|^2_{s-2} \leq 0\,.$
2. $\|w(t)\|^2_{s-2} = e^{-2at}\|\phi\|^2_{s-2} - 2b\int_0^t e^{-2a(t-r)}\|\partial_x w(r)\|^2_{s-2}\,dr \leq e^{-2at}\|\phi\|^2_{s-2}$ *for* $t \geq 0\,.$
3. $\|w(t)\|_{s-2} \leq e^{-at}\|\phi\|_{s-2} \leq e^{-at}\|\phi\|_s \leq \|\phi\|_s$ *for* $t \geq 0\,.$ *That is,* $\lim_{t\to+\infty}\|w(t)\|_{s-2} = 0\,.$

If the external source to the non homogeneous problem decayed uniformly, we will obtain the existence of global solution and its uniform decay.

Theorem 2.3. *Let $s \in R$ and $\varphi \in H^s_{per}$, if $F \in C([0,\infty), H^s_{per})$ and satisfies $\exists\,\mu > 0$, $\mathcal{C} > 0$ such that $\|F(t)\|_s \leq \mathcal{C}e^{-\mu t}$ for all $t \geq 0$. Then there exists a global solution v of (P^F) that satisfies: $\exists\eta > 0$ and $\mathrm{Q}(t)$ polynomial with $\partial\mathrm{Q} \leq 1$ and $\|v(t)\|_s \leq e^{-\eta t}\mathrm{Q}(t)$, $\forall t \geq 0$.*

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